Correspondence between the temporal and spacial behavior in reversible one-dimensional cellular automata equivalent with the full shift

Juan Carlos Seck Tuoh Mora*
Centro de Investigación Avanzada en Ingeniería Industrial
Universidad Autónoma del Estado de Hidalgo
Ciudad Universitaria, Carr. Pachuca-Tulancingo Km 4.5 Pachuca, Hidalgo 42184, México

Genaro Juárez Martínez†
Departamento de Ingeniería Eléctrica, Sección Computación
CINVESTAV-IPN.
Av IPN 2508, Col San Pedro Zacatenco, México D.F. 07360, México

Harold V. McIntosh ‡
Departamento de Aplicación de Microcomputadoras
Instituto de Ciencias, Universidad Autónoma de Puebla
Apartado postal 461, 72000 Puebla, Puebla, México

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Abstract

We present the basic properties of reversible one-dimensional cellular automata equivalent by permutations with the full shift, this work only takes reversible automata of neighborhood size 2. In these cases, we prove that the evolution rule defining the temporal behavior of the automaton may specify the spacial behavior as well. Based in this result we present a procedure for constructing configurations with a predefined dynamical behavior. Some examples of these results are presented.

1 Introduction

Cellular automata were invented by John von Neumann for studying and implementing a self-reproducing system [12]. Important stages in cellular automata theory are provided by John Conway and the automaton *LIFE* [3] and by Stephen Wolfram and his analysis of one-dimensional cellular automata with two states and three neighbors [13].

*jseck@ciaii.edu.mx, juanseck@yahoo.com.mx
†genaro@enigma.red.cinvestav.mx, genarojm@correo.unam.mx
‡mcintosh@servidor.unam.mx, http://delta.cs.cinvestav.mx/˜mcintosh

One special type of cellular automata is the one where the global behavior is invertible, these cellular automata are called *reversible*. The theory of reversible cellular automata has been widely developed for the one-dimensional case, in this sense an outstanding work is developed by Hedlund which establishes fundamental properties of these systems [4]. In particular, Hedlund provides a complete local characterization for the behavior of reversible automata by means of two properties: uniform multiplicity of ancestors and Welch indices. Other important works about reversible one-dimensional cellular automata are developed by Amoroso and Patt [1], Toffoli and Margolus [10, 11], Nasu [8] and Kari [5].

In this paper we shall only study reversible automata of neighborhood size 2 in both rules, and from these ones we will take those having a behavior equivalent with the full shift, where *equivalent* means that there exists a state permutation transforming the automaton into the full shift. For this kind of reversible automata, we shall prove that the evolution rule applied in the temporal direction may be also applied in the spacial one, and we use this result for controlling the dynamical behavior of a given cell in the automaton.

The paper is organized as follows, Section 2 presents the basic properties of reversible one-dimensional cellular automata and defines a reversible automaton equivalent with the full shift by a permutation. Section 3 exposes the correspondence between the temporal and the spacial behavior in reversible automata equivalent with the full shift. Section 4 describes a procedure to specify initial configurations with a cell having a desired dynamical behavior for reversible automata equivalent with the full shift. Section 5 illustrates the results of this paper with two examples and the final section provides the concluding remarks of the paper.

2 Properties of reversible one-dimensional cellular automata equivalent with the full shift

Before defining a one-dimensional cellular automaton, we present one important remark used through this paper. For a finite set K of elements and $m \in \mathbb{Z}^+$, let K^m be the set of sequences such that each $w \in K^m$ is formed by m elements of K; and let K^* be the whole set of finite sequences formed by elements of K. Now we define a cellular automaton.

A one-dimensional cellular automaton $\mathcal{A} = \{k, m, \varphi\}$ consists of a finite set of states K whose cardinality is represented by $k \in \mathbb{Z}^+$, an initial configuration c which is a one-dimensional array of cells where each takes a single element of K, a neighborhood size $m \in \mathbb{Z}^+$ and a mapping $\varphi : K^m \to K$ called the evolution rule of $\mathcal{A}$; in this paper we only use finite configurations.

The dynamics of $\mathcal{A}$ is given as follows, the initial and the final cell of the initial configuration c are concatenated so that c forms a ring, if c has $n \in \mathbb{Z}^+$ cells then we can index every cell of c by $i \in \{0 \dots n-1\}$, the cell of c at position i is represented by c_i. For every c_i we shall take the block of cells $b = c_i \dots c_{i+m-1}$, the states of the cells in b defines a sequence $w \in K^m$; over this sequence we apply the evolution rule $\varphi(w) \to a \in K$. Thus the evolution rule φ produces a new configuration c' with the same number of cells that c and the state of c'_i is produced applying the evolution rule φ over the states of the block $b = c_i \dots c_{i+m-1}$, and c'_i is vertically placed below the block b (representing the temporal evolution) and horizontally placed in the middle of b (representing the spacial colocation), that is c'_i is placed with regard of c_i at position $(i+m-1)/2$.

Now we shall give two relevant definitions in cellular automata theory, for $w \in K^m$ and $a \in K$, if $\varphi(w) = a$ then w is an ancestor of a; we can extend this concept for larger sequences, for $w \in K^n, n \in \mathbb{Z}^+, n \geq m$, let us take $\varphi(w)$ as the application of the evolution rule over each neighborhood forming w, thus $\varphi(w) = v \in K^{n-m+1}$ and w is an ancestor of v.

Given a cellular automaton $\mathcal{A}$, it may be possible that there exists a sequence $w \in K^*$ such that it can not be produced as the evolution of another sequence of states, and w can only appear within the initial configuration, in this case w belongs to the Garden of Eden of $\mathcal{A}$.

We are interested in studying reversible automata, that is, cellular automata whose global behavior is invertible. A one-dimensional cellular automaton $\mathcal{A} = \{k, m, \varphi\}$ is reversible if there exists another one-dimensional cellular automaton $\mathcal{A}^{-1} = \{k, m', \phi\}$ such that $\mathcal{A}^{-1}$ has the inverse global behavior of $\mathcal{A}$, that is, we can take the automaton $\mathcal{A}$ and make it evolve for n steps obtaining a final configuration c', then we can apply $\mathcal{A}^{-1}$ over c' and yield the inverse dynamical behavior up to reach the initial configuration c of $\mathcal{A}$ and moreover we can obtain an ancestor for the initial configuration c (although it sounds as a contradiction for the adjective *initial*).

In this way, a cellular automaton $\mathcal{A}$ is reversible if there are no sequences in the Garden of Eden, that is, every sequence has at least one ancestor. For these systems Hedlund in [4] defines two important local properties presented as follows:

Property 1 (Uniform multiplicity of states) *For a reversible one-dimensional cellular automaton $\mathcal{A} = \{k, m, \varphi\}$, every sequence of states $w \in K^*$ has k ancestors.*

Property 2 (Welch indices) *For a reversible one-dimensional cellular automaton $\mathcal{A} = \{k, m, \varphi\}$ with inverse $\mathcal{A}^{-1} = \{k, m', \phi\}$ and $n \in \mathbb{Z}^+$, $n \geq m'$; every sequence of states $v \in K^n$ has k ancestors $\{w_1 \dots w_k\} \subset K^{n+m-1}$ in $\mathcal{A}$. These ancestors define three sets, a subset $W_L \subset K^{n_1}$ with cardinality L, a unique sequence $u \in K^{n_2}$ and a subset $W_R \subset K^{n_3}$ with cardinality R such that $n_1 + n_2 + n_3 = n + m - 1$, the cartesian product $W_L \times u \times W_R = \{w_1 \dots w_k\}$ and $LR = k$. W_L is a left Welch subset, W_R is a right Welch subset and the values L and R are the Welch indices of $\mathcal{A}$.*

Nasu in [8] also defines another important property which characterizes the contents of W_L and W_R:

Property 3 (Intersection property) *For a reversible one-dimensional cellular automaton $\mathcal{A} = \{k, m, \varphi\}$ and every pair of Welch subsets (W_L, W_R), we have that $W_L \cap W_R = a \in K$.*

We shall only take reversible automata $\mathcal{A} = \{k, 2, \varphi\}$ equivalent with the full shift of k symbols by a given permutation, in particular we shall take the right full shift. A right full shift of k symbols consists of a set of symbols K with cardinality k and an initial array of states c where for $i \in \mathbb{Z}$, each position c_i has assigned a symbol of K. This position is shifted to the right in each time step of the system, time advances in discrete steps; thus we have a new array c' where $c'_{i+1} = c_i$ [9].

A right full shift of k symbols may be simulated by a one-dimensional cellular automaton $\mathcal{A} = \{k, 2, \varphi\}$ [6, 7, 2], for any $a, b \in K$, the evolution rule must hold that $\varphi(a, b) = a$. For the configuration c' produced by the evolution of another configuration c we shall take that $\varphi(c_i, c_{i+1}) = c'_{(i+1/2)}$. A cellular automaton $\mathcal{A} = \{k, 2, \varphi\}$ is equivalent with the full shift of k symbols by a permutation if there exists a permutation $\pi : K \to K$ such that for any $a, b, c \in K$ and $\varphi(a, b) = c$ we have that $\pi(c) = a$. Cellular automata $\mathcal{A} = \{k, 2, \varphi\}$ equivalent with the full shift of k symbols by a permutation are characterized in the following way:

1. The evolution rule φ can be presented as a matrix M_φ; each row index is a state $a \in K$, each column index is a state $b \in K$ and the entry $(a, b) \in M_\varphi$ is equal to $\varphi(a, b) = c \in K$.
2. All the elements of a given row in M_φ are equal to some state of K.

3. All the columns of M_φ are equal to the same permutation of states in K.

Based on the properties of uniform multiplicity, Welch indices and the intersection of Welch subsets, in the next section we shall describe the equivalence between the temporal and the spacial behavior of automata equivalent with the full shift.

3 Correspondence between the temporal and the spacial behavior in reversible automata equivalent with the full shift

A reversible automaton $\mathcal{A} = \{k, 2, \varphi\}$ equivalent with the full shift of k symbols by a permutation is a special type of reversible one-dimensional cellular automaton with Welch indices $L = 1$ and $R = k$, that is, the left part of every neighborhood defines its evolution. Since the inverse behavior of a right full shift is a left full shift with the same number of symbols, and for any permutation $\pi : K \to K$ there exists the permutation $\pi^{-1} : K \to K$ such that $\pi^{-1}(\pi(a)) = a \in K$, then for the automaton $\mathcal{A}$ there exists the inverse automaton $\mathcal{A}^{-1} = \{k, 2, \phi\}$ with the inverse behavior and the following properties:

1. The evolution rule ϕ can be presented as a matrix M_ϕ; each row index is a state $a \in K$, each column index is a state $b \in K$ and the entry $(a, b) \in M_\phi$ is equal to $\phi(a, b) = c \in K$.
2. All the elements of a given column in M_ϕ are equal to some state of K.
3. All the rows of M_ϕ are equal to the same permutation of states in K.
4. For $a, b \in K$ we have that $\pi(\phi(a, b)) = b$.

In this way the inverse automaton $\mathcal{A}^{-1}$ has Welch indices $L^{-1} = k$ and $R^{-1} = 1$ and the same neighborhood size than $\mathcal{A}$, therefore we can use the properties of reversible automata for characterizing the evolution of $\mathcal{A}$ and $\mathcal{A}^{-1}$.

1. For both $\mathcal{A}$ and $\mathcal{A}^{-1}$, every state $a \in K$ has k ancestors .
2. For $\mathcal{A}$, the ancestors of every $a \in K$ start from a single state and end into $R = k$ distinct states.
3. For $\mathcal{A}^{-1}$, the ancestors of every $a \in K$ start from $L^{-1} = k$ different states and end into a single state.

Thus for a given state $a \in K$, the number of ancestors defined from both $\mathcal{A}$ and $\mathcal{A}^{-1}$ are described in Figure 1, in this figure the contents of the ancestor cells at both directions illustrate the number of possible states in each cell.

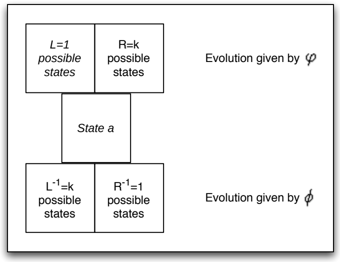

Figure 1: Form and number of ancestors for any $a \in K$ defined by $\mathcal{A} = \{k, 2, \varphi\}$ and $\mathcal{A}^{-1} = \{k, 2, \phi\}$.

Figure 1 illustrates the behavior of the Welch indices for this kind of automata, we can see that $R = L^{-1}$ and $L = R^{-1}$, in this way we can specify other evolution rules φ' and ϕ' (Figure 2).

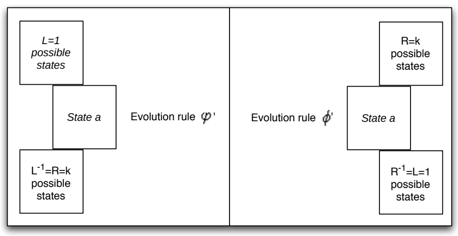

Figure 2: Evolution rules φ' and ϕ' induced by the Welch indices of $\mathcal{A}$ and $\mathcal{A}^{-1}$.

We can easily show that φ' and ϕ' also define reversible one-dimensional cellular automata, first we take the evolution rule φ':

- Each state has $LL^{-1} = LR = k$ ancestors fulfilling with Property 1.
- The ancestors of each state are divided in two parts, one with $L = 1$ possible different states and another with $L^{-1} = R = k$ states, fulfilling with Property 2. These two parts define Welch subsets W_L and $W_{L^{-1}}$.
- Since $L = 1$ and $L^{-1} = R = k$, we have that $W_L \cap W_{L^{-1}} = a \in K$ and this intersection is unique, holding with Property 3. This property makes extensive the previous properties for every sequence $w \in K^*$.

The previous properties are analogous for ϕ'. The relevant remark in this part is that $L^{-1} = R$ for every $a \in K$, therefore $W_{L^{-1}} = W_R$ and $\varphi = \varphi'$, analogously $R^{-1} = L$ and $W_{R^{-1}} = W_L$ hence $\phi = \phi'$. In this way φ specifies the temporal evolution of the automaton but also defines a spacial evolution going from left to right, and ϕ defines the inverse temporal evolution of the automaton and establishes as well another spacial evolution going now from right to left. We shall use this feature in the following section in order to describe a procedure for generation initial configurations of $\mathcal{A}$ with a desired dynamical behavior.

4 Procedure for constructing initial configurations with a desired dynamical behavior

Suppose that we have a reversible automaton $\mathcal{A} = \{k, 2, \varphi\}$ equivalent with the full shift of k symbols by a permutation, and we desire that c_i in the initial configuration takes predefined values as $\mathcal{A}$ evolves.

By the properties of $\mathcal{A}$ we know that there exists another $\mathcal{A}^{-1} = \{k, 2, \phi\}$ such that φ defines some possible spacial evolution from left to right and ϕ defines another spacial evolution in the inverse direction. Thus in order to achieve that c_i takes a list of predefined values we shall specify the next procedure:

Procedure 1 (Controlling the dynamical behavior of c_i)

1. *Take the list of $n \in \mathbb{Z}^+$ predefined values for c_i and place them in the desired order into their corresponding positions within the evolution space. For $j \in \mathbb{N}$ enumerate these cells as c_i^{2j} where the superscript indicates that the cell is located into the $2j$-th generation.*
2. *For each pair $c_i^{2j}, c_i^{2(j+1)}$, apply both $\varphi(c_i^{2j}, c_i^{2(j+1)}) = a \in K$ and $\phi(c_i^{2j}, c_i^{2(j+1)}) = b \in K$; place a and b into the right and the left cell respectively between c_i^{2j} and $c_i^{2(j+1)}$. This step generates two new lists of cells, one for the states produced on the right of the cells c_i^{2j} and another for the states yielded on the left of these cells.*
3. *For the right list produced in the previous step, apply step 2 only using the evolution rule φ and placing the resulting cells into the right side of the list. For the left list yielded in the previous step, apply step 2 using only ϕ and placing the resulting cells into the left side of this list. In the case of lists produced in odd steps, take both lists as a ring concatenating the last element with the first one for generating new lists with the same number of cells and positions that the original one. The procedure stops after $2(n-1)$ iterations.*

In the next section we present two examples using this process, the first with five states and the second with six states.

5 Illustrative examples

5.1 Automaton $\mathcal{A} = \{5, 2, \varphi\}$

Take the matrix M_φ for the evolution rule of the automaton $\mathcal{A}$ (Table 1):

	0	1	2	3	4
0	4	4	4	4	4
1	2	2	2	2	2
2	0	0	0	0	0
3	1	1	1	1	1
4	3	3	3	3	3

Table 1: Matrix M_φ for the automaton $\mathcal{A} = \{5, 2, \varphi\}$.

The permutation specifying the equivalence of the automaton with the full shift is $\pi = \{0 \to 4, 1 \to 2, 2 \to 0, 3 \to 1, 4 \to 3\}$, hence the inverse permutation is $\pi^{-1} = \{0 \to 2, 1 \to 3, 2 \to 1, 3 \to 4, 4 \to 0\}$ and the inverse evolution rule ϕ has the following specification:

	0	1	2	3	4
0	2	3	1	4	0
1	2	3	1	4	0
2	2	3	1	4	0
3	2	3	1	4	0
4	2	3	1	4	0

Table 2: Matrix M_ϕ for the automaton $\mathcal{A}^{-1} = \{5, 2, \phi\}$.

Suppose that the list of desired states for the cell c_i is $l = \{4, 2, 0, 3, 2\}$, then we shall use Procedure 1 in order to calculate an initial configuration where c_i evolves into the states of l, the steps of this procedure are described in Figure 3.

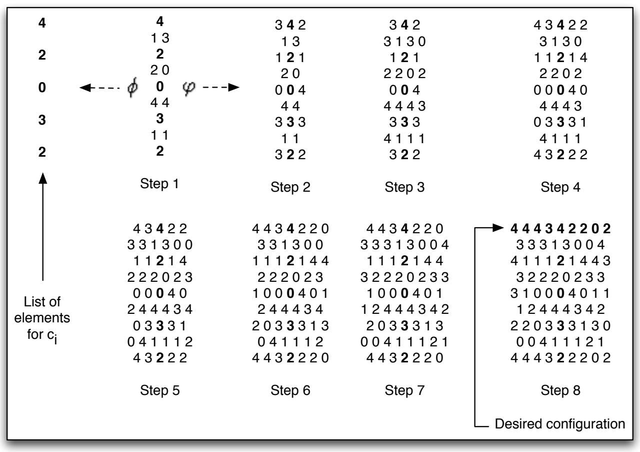

Figure 3: Initial configuration calculated by Procedure 1.

In this way the configuration 444342202 evolves into the desired states for the cell c_i, Figure 4 shows the evolution of this configuration taking it as a ring.

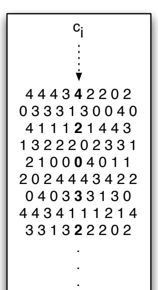

Figure 4: Evolution of the configuration 444342202 generating the desired behavior for c_i, note that this evolution is different from the one obtained in Figure 3.

5.2 Automaton $\mathcal{A} = \{6, 2, \varphi\}$

Take the matrix M_φ in Table 3 representing the evolution rule of $\mathcal{A}$, in this case the permutation defining the equivalence of $\mathcal{A}$ with the full shift $\pi = \{0 \to 3, 1 \to 0, 2 \to 4, 3 \to 1, 4 \to 5, 5 \to 2\}$.

	0	1	2	3	4	5
0	3	3	3	3	3	3
1	0	0	0	0	0	0
2	4	4	4	4	4	4
3	1	1	1	1	1	1
4	5	5	5	5	5	5
5	2	2	2	2	2	2

Table 3: Matrix M_φ for the automaton $\mathcal{A} = \{6, 2, \varphi\}$.

The inverse permutation of π is $\pi^{-1} = \{0 \to 1, 1 \to 3, 2 \to 5, 3 \to 0, 4 \to 2, 5 \to 4\}$, this permutation defines the matrix M_ϕ corresponding with the automaton $\mathcal{A}^{-1} = \{6, 2, \phi\}$ (Table 4) .

	0	1	2	3	4	5
0	1	3	5	0	2	4
1	1	3	5	0	2	4
2	1	3	5	0	2	4
3	1	3	5	0	2	4
4	1	3	5	0	2	4
5	1	3	5	0	2	4

Table 4: Matrix M_ϕ for the automaton $\mathcal{A}^{-1} = \{6, c, 2, \phi\}$.

Let us take the list $l = \{0, 5, 3, 2, 4\}$ for the predefined states of c_i, then apply Procedure 1 we have the result presented in Figure 5.

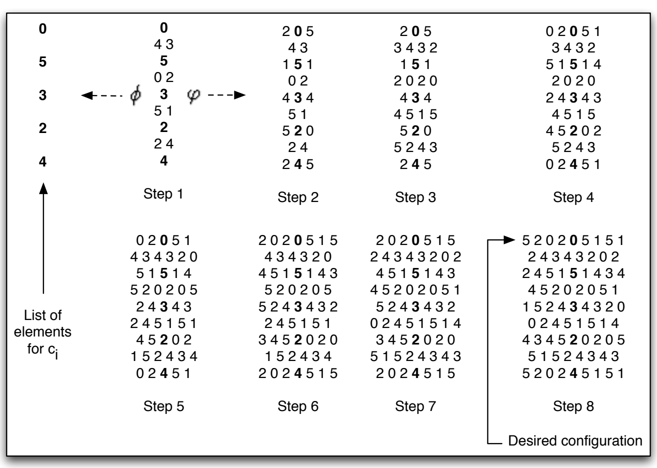

Figure 5: Initial configuration obtained by Procedure 1.

Thus the initial configuration 520205151 evolves into the predefined states for c_i, Figure 6 depicts the evolution of this configuration taking it as a ring.

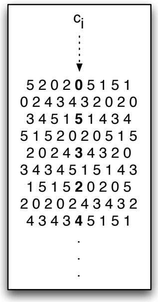

Figure 6: Evolution of 520205151 yielding the predefined states for c_i.

6 Concluding remarks

Cellular automata equivalent with the full shift have a very particular behavior characterized by the form of their Welch subsets and the neighborhood size of their evolution rules. In this kind of automata we can observe that the same dynamics can be defined in the temporal and spacial sense.

A further work is trying to extend these results for any reversible one-dimensional cellular automata, this extension is in function of the features of the Welch indices, thus if we desire to know when a reversible automaton has the same temporal and spacial behavior, then we have to characterize its Welch subsets.

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