Unconventional invertible behaviors in reversible one-dimensional cellular automata

(Short title: Unconventional behaviors in reversible automata)

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Abstract

Reversible cellular automata are discrete invertible dynamical systems determined by local interactions among their components. For the one-dimensional case, there are classical references providing a complete characterization based on combinatorial properties. Using these results and the simulation of every automaton by another with neighborhood size 2, this paper describes other types of invertible behaviors embedded in these systems different from the classical one observed in the temporal evolution. In particular spatial reversibility and diagonal surjectivity are studied, and the generation of macrocells in the evolution space is analyzed.

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1 Introduction

Cellular automata are discrete dynamical systems in time and space, whose behavior is based on local interactions defined among their components. These systems were proposed by John von Neumann for proving the possibility of constructing self-reproducing machines [von Neumann, 1966]. Since this original work, several issues and variations of the model have been studied, one of this topics is the specification, characterization and enumeration of reversible cellular automata. The complexity in these systems is given by the non-invertible local mappings among their components which induce a global reversible behavior; therefore their study has been widely developed both for understanding relevant theoretical questions and for practical applications in several fields in physics, chemistry, biology and engineering [Wolfram,1986], [Toffoli & Margolus,1987], [Toffoli & Margolus, 1991], [Kari, 1992a], [Chopard & Droz, 1998], [Adamatzky, 2003].

Classical references in the theoretical analysis are the papers by Myhill [1963] and Moore [1970] investigating Garden-of-Eden structures which cannot be produced by the local interactions of a cellular automaton and, by Amoroso and Patt [1972] giving a first computational procedure for enumerating reversible automata; this line of research has been followed in several works [Hillman, 1991], [Moraal, 2000], [Boykett, 2004], [Seck-Tuoh *et al.*, 2005].

A complete local characterization of reversible cellular automata is developed for the one-dimensional case by Hedlund [1969] and Nasu [1978] using a combinatorial, topological and graph-theoretical approach. In these systems, reversibility is given in the temporal sense; Boykett [2003], Hillman [2004] and Boykett [2006] have studied reversible automata from an algebraic perspective, some of their results suggest the existence of other types of reversible behaviors different from the classical one.

This paper applies the results defined by Hedlund and Nasu in order to characterize reversible behaviors distinct from the temporal one in one-dimensional cellular automata, in particular we expose properties for obtaining spatially reversibility, formation of macrocells and diagonal surjective dynamics. These results are generalized by modeling every one-dimensional cellular automaton by another with neighborhood size 2, thereby we just need to take this kind of automata to understand the other cases.

This work is an extension of our research about reversible cellular automata [Seck-Tuoh, 2002], [Seck-Tuoh *et al.*, 2003], [Seck-Tuoh *et al.*, 2006], in particular improving the conclusions obtained in [Seck-Tuoh *et al.*, 2004]; the intention of this manuscript falls into the theoretical and experimental study, expecting that other researches may find some helpful conclusions which may be applied both for academic and practical interests.

The paper is organized as follows: Sec. 2 gives the basic concepts of one-dimensional cellular automata; Sec. 3 explains the representation of any cellular automaton by another with neighborhood size 2 for simplifying the study and presents the local properties for the reversible case. Section 4 describes both spatial reversibility and the composition of macrocells in the evolution space; Sec. 5 exposes distinct diagonal surjective behaviors in reversible automata and Sec. 6 exposes the concluding remarks of the work. For clarification, illustrative examples are provided during the progression of the paper.

2 One-Dimensional Cellular Automata

A cellular automaton $A = \{m_s, m_n, \varphi\}$ consists of a finite set of states S with $|S| = m_s$ and neighborhood size m_n establishing an evolution rule $\varphi : S^{m_n} \rightarrow S$. For any $m \in \mathbb{Z}^+$ let $\mathbb{Z}_m$ be the set of integers *mod m*. Thus given $m_c \in \mathbb{Z}^+$, an initial configuration $c^0 : \mathbb{Z}_{m_c} \rightarrow S$ is provided; if time is understood, a configuration will be only defined by c. Let c_i be the cell at position $i \, mod \, m_c$ in c and let $\gamma(c_i)$ be the state at cell c_i.

In this paper periodic boundary conditions are taken concatenating c_{m_c-1} to c_0. Let $n_i = \gamma(c_i) \ldots \gamma(c_{i+m_n-1})$ be a neighborhood of the automaton; hence we have m_c neighborhoods in c and neighborhoods n_i and n_{i+1} overlap in $m_n - 1$ states.

Time advances in discrete steps, the dynamics of A is given applying the evolution rule over all the neighborhoods in c^t, where $\varphi(n_i^t) = \gamma(c_i^{t+1})$; this state is centered below n_i^t. Thusly a new configuration $c^{t+1} : \mathbb{Z}_{m_c} \to S$ is obtained and φ induces a global mapping $\Phi : S^{m_c} \to S^{m_c}$.

Definition of the evolution rule can be extended for larger sequences of states; for each $m \geq m_n$ and every $v \in S^m$, $\varphi(v) = w \in S^{m-m_n+1}$ applying φ over all the neighborhoods forming v.

The evolution rule is represented by a matrix M_φ where rows and columns indices are the sequences in S^{m_n-1}; entry $(s_1w, ws_2) = s_3$ in M_φ if $\varphi(s_1ws_2) = s_3$ with $w \in S^{m_n-2}$ and $s_i \in S$ for $1 \leq i \leq 3$. Matrix M_φ will be useful to describe non-classical reversible and surjective behaviors.

3 Properties of Reversible Automata

Before presenting the properties of reversible one-dimensional cellular automata, a relevant result established independently by Kari [1992b] and Boykett [1997] will be explained. This deduction is worthwhile to simplify the study in the rest of the paper.

For any $w_1, w_2 \in S^{m_n-1}$ it is hold that $\varphi(w_1w_2) \in S^{m_n-1}$; in this way the evolution rule determines a mapping $\varphi : S^{2m_n-2} \to S^{m_n-1}$. Let us take a new set of states U such that $|U| = m_u = m_s^{m_n-1}$, hence a bijection $\alpha : S^{m_n-1} \to U$ can be defined such that $\upsilon = \alpha \circ \varphi \circ \alpha^{-1}$ and $\upsilon : U^2 \to U$. Figure 1 presents the case for a neighborhood size 3.

Thus any cellular automaton $A = \{m_s, m_n, \varphi\}$ can be simulated by another $A' = \{m_u, 2, \upsilon\}$; therefore in the rest of this paper we shall only analyze cellular automata with neighborhood size 2 because the other cases can be represented by this one. In particular, rows and columns indices in M_υ are the states in U and each entry $(u_1, u_2) = u_3$ iff $\upsilon(u_1u_2) = u_3$ for $u_i \in U$,

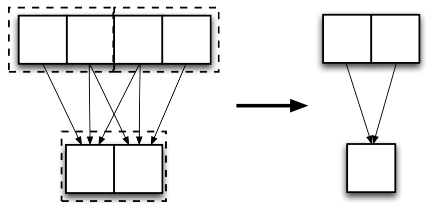

Figure 1: A neighborhood size 3 simulated by a neighborhood size 2.

$1 \leq i \leq 3$.

A cellular automaton is reversible if its global behavior is reversed by the action of another cellular automaton (the inverse one). Since each can be simulated by another with neighborhood size 2, we can conclude that both automata hold that $m_n = 2$.

Formally a reversible automaton is defined by $A_R = \{m_s, 2, \varphi, l_\varphi, r_\varphi\}$ and its inverse is analogously described by $A_R^{-1} = \{m_s, 2, \varphi^{-1}, l_{\varphi^{-1}}, r_{\varphi^{-1}}\}$ such that for any $c \in S^{m_c}$, if $\Phi(c) = c'$ in A_R then $\Phi^{-1}(c') = c$ in A_R^{-1}. Definition of A_R has two additional parameters l_φ and r_φ known as *Welch indices*, these invariant positive integers will be useful for describing the local properties of these systems; in order to facilitate their explanation, some definitions about the preimages of finite sequences are given.

For $m \in \mathbb{Z}^+$ and every $w \in S^{m-1}$, let $\Lambda(w) = \{v : v \in S^m, \varphi(v) = w\}$ be the set of ancestors of w. For every $w \in S^m$, let $L_\varphi(w) = \{s \in S \mid \exists\, v \in K^m$ such that $sv \in \Lambda(w)\}$ be the left Welch set of w with regard of φ; analogously let $R_\varphi(w) = \{s \in S \mid \exists\, v \in S^m$ such that $vs \in \Lambda(w)\}$ be the right Welch set of w. Reversible automata are fully characterized by Hedlund [1969] and Nasu [1978]; in particular demonstrating the following three properties:

Property 1 *For every $m \in \mathbb{Z}^+$ and each $w \in K^m$, $|\Lambda(w)| = m_s$.*

Property 2 *For every $m \in \mathbb{Z}^+$ and each $w \in K^m$, $|L_\varphi(w)| = l_\varphi$, $|R_\varphi(w)| = r_\varphi$ and $l_\varphi r_\varphi = m_s$.*

Property 3 *For every* $m \in \mathbb{Z}^+$ *and any* $w_1, w_2 \in S^m$, $|R_\varphi(w_1) \cap L_\varphi(w_2)| = 1$.

These features define as well a particular structure for M_φ when the reversible automaton and its inverse have neighborhood size 2; in this case every state appears in m_s entries of M_φ, moreover each state occurs at the same r_φ positions in exactly l_φ rows.

In this way for a reversible automaton A_R we can define a transpose automaton A_R^T with evolution rule φ^T holding that $M_{\varphi^T} = M_\varphi^T$. Since the previous properties show that the behavior of reversible automata depends on the cardinality and elements of Welch sets but not on their position, a straightforward conclusion is that A_R^T is also reversible. Based on these results, Secs. 4 and 5 expose unconventional bijective and surjective behaviors in reversible automata.

4 Spatial Reversibility

Classically in a cellular automaton the invertible behavior is viewed in the temporal sense, where the transition from c^t into c^{t+1} is reversible. Based on the properties previously exposed, another invertible behavior is described in the spatial direction.

Definition 1 *A reversible cellular automaton* A_R *is spatially reversible if there exists another reversible automaton* $A_T = \{m_s, 2, \tau, l_\tau, r_\tau\}$ *such that for every* $i \in \mathbb{Z}_{m_c}$ *and each* $t \in \mathbb{Z}$ *it is fulfilled that* $\tau(\gamma(c_{i+1}^{t-1})\gamma(c_i^{t+1})) = \tau^{-1}(\gamma(c_i^{t-1})\gamma(c_{i-1}^{t+1}) = \gamma(c_i^t)$ *(Fig. 2).*

In order to analyze spatial reversibility, additional features about the cardinality of Welch sets are described; some of them using the spatial and temporal coordinates of cells in a given configuration. Let us define $\Gamma(c_i^t) \subseteq S$ as the set of possible states assigned to c_i^t.

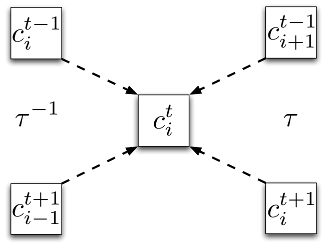

Figure 2: Cells defining spatial reversibility.

For each $s \in S$ and any state $q_2 \in R_\varphi(s)$, it is fulfilled that $\varphi(q_1 q_2) = s$ for every $q_1 \in L_\varphi(s)$ in order to hold with Property 1. In this way by Property 2, for a fixed $\gamma(c_i^t)$ we have that $|\Gamma(c_i^{t-1})| = l_\varphi$ and $|\Gamma(c_{i+1}^{t-1})| = r_\varphi$. Taking now φ^{-1}, $|\Gamma(c_{i-1}^{t+1})| = l_{\varphi^{-1}}$ and $|\Gamma(c_i^{t+1}| = r_{\varphi^{-1}}$.

Let us set $\Gamma(c_{i-1}^t) = S$ and a fixed $\gamma(c_i^t)$, hence by Property 2 $|\Gamma(c_{i-1}^{t+1})| = r_\varphi$; analogously if we assign $\Gamma(c_{i+1}^t) = S$ we have that $|\Gamma(c_i^{t+1})| = l_\varphi$. Therefore $l_{\varphi^{-1}} = r_\varphi$ and $r_{\varphi^{-1}} = l_\varphi$.

The last observation implies that for a fixed $\gamma(c_i^t)$, $|\Gamma(c_{i+1}^{t-1})| = r_\varphi$ and $|\Gamma(c_i^{t+1})| = l_\varphi$; therefore these sets hold with Properties 1 and 2. Thus only two additional features must be considered to know whether the automaton is spatially reversible:

- Reviewing Property 3.
- Checking that τ and τ^{-1} are well-defined rules [1].

This revision can be executed applying the following two conditions:

1. For any $q_1, q_2 \in S$, $|L_\varphi(q_1) \cap L_{\varphi^{-1}}(q_2)| = 1$ and $|R_\varphi(q_1) \cap R_{\varphi^{-1}}(q_2)| = 1$.
2. For any $q_1, q_2 \in S$, there are not $q_3, q_4 \in S$ such that:
   - $q_3 \in L_\varphi(q_1) \cap L_\varphi(q_2)$ and $q_4 \in L_{\varphi^{-1}}(q_1) \cap L_{\varphi^{-1}}(q_2)$

[1] That is, that every state has m_s preimages in both rules

or

- $q_3 \in R_\varphi(q_1) \cap R_\varphi(q_2)$ and $q_4 \in R_{\varphi^{-1}}(q_1) \cap R_{\varphi^{-1}}(q_2)$

Condition 2 examines if there are two states sharing the same ancestor neighborhood in τ and τ^{-1}, opposite to the definition of a cellular automaton. The above conditions provide an algorithm $O(m_s^3)$ for inspecting spatial reversibility.

Supplementary results can be determined depending on the initial specification of Welch sets. Definition 1 arbitrary specifies for A_T that left cells of neighborhoods are those placed at time $t-1$ both for τ and τ^{-1}; in this way for every $s \in S$ we have that $L_\tau(s) = R_\varphi(s)$ and $R_\tau(s) = R_{\varphi^{-1}}(s)$. As a consequence for a reversible automaton A_R, if $L_\varphi(s) = R_{\varphi^{-1}}(s)$ hence $M_\tau = M_\varphi^T$; therefore $A_T = A_R^T$. Analogously $L_{\tau^{-1}}(s) = L_\varphi(s)$ and $R_{\tau^{-1}}(s) = L_{\varphi^{-1}}(s)$; consequently if $R_\varphi(s) = L_{\varphi^{-1}}(s)$ thence $\tau^{-1} = \varphi$ and $A_T^{-1} = A_R$. For instance, take $A_R = \{4, 2, \varphi, 2, 2\}$ where matrices M_φ, $M_{\varphi^{-1}}$ and their respective Welch sets are illustrated in Table 1.

M_φ		$M_{\varphi^{-1}}$		Welch sets				
	0123		0123	State s	$L_\varphi(s)$	$R_\varphi(s)$	$L_{\varphi^{-1}}(s)$	$R_{\varphi^{-1}}(s)$
0	1010	0	1133	0	{0,1}	{1,3}	{1,2}	{0,1}
1	1010	1	0022	1	{0,1}	{0,2}	{0,3}	{0,1}
2	2323	2	0022	2	{2,3}	{0,2}	{1,2}	{2,3}
3	2323	3	1133	3	{2,3}	{1,3}	{0,3}	{2,3}

Table 1: Evolution rule and Welch sets of an spatially reversible automaton.

Table 1 shows that for every $s \in S$, $L_\varphi(s) = R_{\varphi^{-1}}(s)$; therefore $A_T = A_R^T$. An example of this spatial reversible behavior is depicted in Fig. 3, τ and τ^{-1} can be identified using M_φ^T. Spatial reversibility is general for any reversible automaton with a Welch index equal to 1; let us take A_R with $l_\varphi = 1$, this implies by Properties 1 and 2 that $r_\varphi = m_s$ leading as well that $l_{\varphi^{-1}} = m_s$ and $r_{\varphi^{-1}} = 1$. This assertion indicates that both $L_\varphi(s) = R_{\varphi^{-1}}(s)$ and $R_\varphi(s) = L_{\varphi^{-1}}(s)$ for each $s \in S$, as result A_R is spatially reversible; in particular $A_T = A_R^T$, $A_T^{-1} = A_R$ in consequence $A_R^{-1} = A_R^T$.

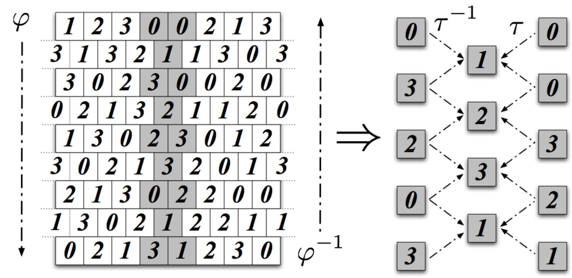

Figure 3: Spatial reversible behavior produced by rules in Table 5.

Spatial reversibility is a conserving process where there is not lost of information in the spatial sense. A concrete case of reversible automata which is not spatially reversible is the one producing the so-called *macrocells* [Brown, 1987] [McIntosh, 1990]; informally they are periodic structures isolated by barriers in the evolution of the automaton.

Welch sets offer a way for detecting non-spatial reversible automata which besides are able to form macrocells; given an automaton A_R if there exists $s \in S$ such that $L_\varphi(s) = L_{\varphi^{-1}}(s)$ and $R_\varphi(s) = R_{\varphi^{-1}}(s)$; a macrocell can be defined. Choose $0 \leq i_1 < i_2 < m_c$ such that $\gamma(c_i^0) = s$ for $i \in \{i_1, i_2\}$. This implies that $\gamma(c_{i-1}^1) \in L_\varphi(s)$ and $\gamma(c_i^1) \in R_\varphi(s)$; accordingly $\gamma(c_{i-1}^2) = s$. In general $\gamma(c_{i-(t+1)}^{2t+1}) \in L_\varphi(s)$, $\gamma(c_{i-t}^{2t+1}) \in R_\varphi(s)$ and $\gamma(c_{i-t}^{2t}) = s$, delimitating the barriers of a macrocell. For instance, take $A_R = \{4, 2, \varphi, 2, 2\}$ in Table 2.

	M_φ		$M_{\varphi^{-1}}$	Welch sets				
	0123		0123	State s	$L_\varphi(s)$	$R_\varphi(s)$	$L_{\varphi^{-1}}(s)$	$R_{\varphi^{-1}}(s)$
0	1221	0	1122	0	{1,3}	{1,2}	{1,3}	{1,2}
1	1001	1	3003	1	{0,1}	{0,3}	{0,2}	{0,1}
2	3223	2	1122	2	{0,2}	{1,2}	{0,2}	{2,3}
3	3003	3	3003	3	{2,3}	{0,3}	{1,3}	{0,3}

Table 2: Reversible automaton able to form macrocells.

In Table 2 we can see that $L_\varphi(0) = L_{\varphi^{-1}}(0)$ and $R_\varphi(0) = R_{\varphi^{-1}}(0)$; in this way a vertical macrocell can be defined using state 0. Figure 4 shows an example of this feature in the evolution of the automaton.

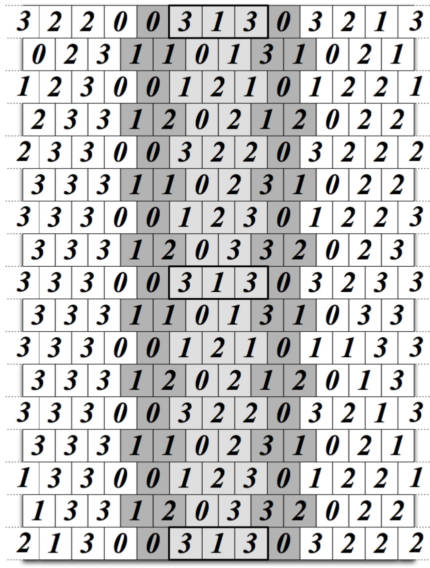

Figure 4: Vertical macrocell (lightgray cells) delineated by state 0 and its Welch sets (dark-gray cells).

5 Diagonal Surjectivity

Other particular case of unconventional surjectivity can be observed on the diagonal direction in the evolution of a reversible automaton, going from the upper-right side to the lower-left one [2].

Definition 2 *A reversible automaton A_R is diagonal surjective if for any $m \in \mathbb{Z}^+$ and $0 \leq j < m$ the diagonal sequence $\gamma(c_{i-j}^{t+j+1})$ and $\gamma(c_{i-m}^{t+m})$ define uniquely the diagonal sequence $\gamma(c_{i-j}^{t+j})$ (Fig. 5).*

A direct result is that every reversible automaton is diagonal surjective; notice that $\gamma(c_{i-m+1}^{t+m-1})$ is uniquely defined by $\varphi^{-1}(\gamma(c_{i-m}^{t+m})\,\gamma(c_{i-m+1}^{t+m})$. From here for all $s \in \Gamma(c_{i-m+2}^{t+m-1})$, the mapping $\varphi^{-1}(\gamma(c_{i-m+1}^{t+m-1}), s)$ yields the same result carrying out with Property 1. Thus $\gamma(c_{i-m+2}^{t+m-2})$

[2] These results can be also applied in the opposite diagonal direction.

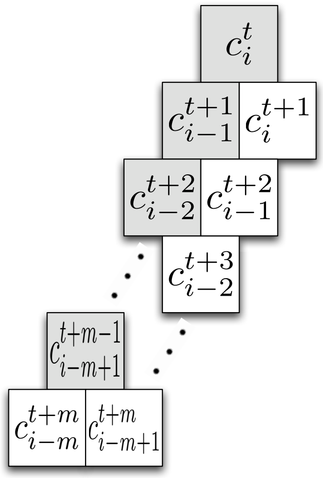

Figure 5: In a diagonal surjective automaton, the state of gray cells is defined by those in the white ones.

is uniquely defined. The generalization of this process establishes the property.

Thus the states in a diagonal sequence depend on the contiguous right ones; in fact the current state establishes the previous one in the desired diagonal. Following this idea, a stronger property can be defined where once specified the m_d first diagonal states, the rest of them are fixed.

Definition 3 *A reversible automaton A_R is m_d-diagonal surjective if for any $m \geq 2$ with $0 \leq j < m$ and each diagonal sequence $\gamma(c_{i-j}^{t+j+1})$ there exists $m_d \in \mathbb{Z}^+$ with $m_d < m$ such that for $1 \leq k \leq m_d$, the sequence of sets $\Gamma(c_{i-m+k-1}^{t+m-k+1})$ defines a unique sequence $\gamma(c_{i-p}^{t+p})$ for $0 \leq p \leq m - m_d$.*

Definiton 3 says that it does not matter what states are in cells $c_{i-m+k-1}^{t+m-k+1}$, the remaining ones in the diagonal are equally specified. In order to complement Definition 3, if for every $m \in \mathbb{Z}^+$ we cannot find some m_d such that the automaton is m_d-diagonal surjective, we say that it is $\aleph$-diagonal surjective. Certain families of reversible automata can be characterized using Definition 3 due to the distinctive attributes of their Welch sets.

The first remark is that every $A_R = \{m_s, 2, \varphi, 1, m_s\}$ is 1-diagonal surjective; observe that for each sequence $\gamma(c_{i-j}^{t+j+1})$ and every state in $\Gamma(c_{i-m}^{t+m})$ it is fulfilled that $|\Gamma(c_{i-j}^{t+j})| = l_\varphi = 1$, then the sequence $\gamma(c_{i-p}^{t+p})$ is uniquely established for $0 \leq p \leq m-1$.

Additional two cases can be typified when both Welch indices are different from 1, one with minimum and another with maximum diagonal surjectivity.

When for every $s \in S$ and for each $\{q_1, q_2\} \subseteq L_\varphi(s)$ it is satisfied that $R_\varphi(q_1) = R_\varphi(q_2)$ hence the automaton is 2-diagonal surjective. For this note that $\gamma(c_{i-m+1}^{t+m-1}) = \varphi(\gamma(c_{i-m}^{t+m})\,\gamma(c_{i-m+1}^{t+m}))$; since $l_\varphi \geq 2$, $|\Gamma(c_{i-m+k}^{t+m-k})| \geq 2$ for $k \in \{0,1\}$. This is why the automaton is at least 2-diagonal surjective.

Seeing that for every $s \in \Gamma(c_{i-m+1}^{t+m-1})$ $R_\varphi(s)$ is the same, so $|\Gamma(c_{i-m+2}^{t+m-2})| = 1$ and consequently $|\Gamma(c_{i-p}^{t+p})| = 1$ for $0 \leq p \leq m-2$, showing that the automaton is 2-diagonal surjective. For instance, take the automaton $A_R = \{4, 2, \varphi, 2, 2\}$ in Table 3.

M_φ		$M_{\varphi^{-1}}$		Welch sets				
	0123		0123	State s	$L_\varphi(s)$	$R_\varphi(s)$	$L_{\varphi^{-1}}(s)$	$R_{\varphi^{-1}}(s)$
0	3300	0	2332	0	{0,2}	{2,3}	{1,3}	{0,3}
1	1122	1	0110	1	{1,3}	{0,1}	{1,3}	{1,2}
2	3300	2	2332	2	{1,3}	{2,3}	{0,2}	{0,3}
3	1122	3	0110	3	{0,2}	{0,1}	{0,2}	{1,2}

Table 3: Evolution rules and Welch sets of a 2-diagonal surjective automaton

Table 3 shows that all states in each left Welch set have identical right Welch sets. As example we shall take a fixed diagonal sequence of 7 cells in Fig. 6 for illustrating the 2-diagonal surjectivity.

The last case is produced when for every state $s \in S$ and each $\{q_1, q_2\} \subseteq L_\varphi(s)$ it is carried out that $R_\varphi(q_1) \cap R_\varphi(q_2) = \varnothing$. In this situation as in the previous one $|\Gamma(c_{i-m+k}^{t+m-k}| \geq 2$ for $k \in \{0,1\}$. Since for every $\{q_1, q_2\} \subseteq \Gamma(c_{i-m+1}^{t+m-1})$ $R_\varphi(q_1) \cap R_\varphi(q_2) = \varnothing$, so $|\Gamma(c_{i-m+2}^{t+m-2})| > 1$

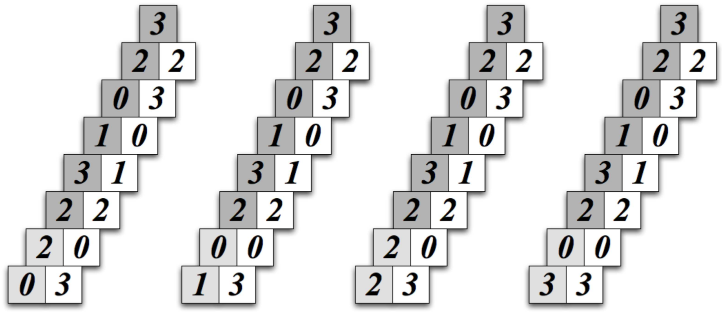

Figure 6: For the diagonal sequence 2301203 and all possible states in the lightgray cells, states in the darkgray ones are the same in the evolution of the automaton.

and the automaton is at least 3-diagonal surjective. The generalization of this process shows that $|\Gamma(c_{i-m+k}^{t+m-k})| > 1$ for $0 \leq k \leq m$ and each $m \in \mathbb{Z}^+$, hence the automaton is $\aleph$-diagonal surjective. For instance, take the automaton $A_R = \{4, 2, \varphi, 2, 2\}$ in 4.

	M_φ		$M_{\varphi^{-1}}$	Welch sets				
	0123		0123	State s	$L_\varphi(s)$	$R_\varphi(s)$	$L_{\varphi^{-1}}(s)$	$R_{\varphi^{-1}}(s)$
0	2323	0	2200	0	{1,2}	{0,2}	{0,2}	{2,3}
1	0303	1	1331	1	{2,3}	{1,3}	{1,3}	{0,3}
2	0101	2	2200	2	{0,3}	{0,2}	{0,2}	{0,1}
3	2121	3	1331	3	{0,1}	{1,3}	{1,3}	{1,2}

Table 4: Evolution rules and Welch sets of a $\aleph$-diagonal surjective automaton.

Table 4 illustrates that in every left Welch set, each state has a different right Welch set. Figure 7 depicts 2 diagonal sequences to exemplify the last case.

6 Concluding Remarks

This manuscript has characterized some unconventional reversible and surjective behaviors based on the properties of Welch sets in one-dimensional reversible cellular automata. These results are relevant for obtaining a better understanding about the dynamical behavior of these systems, the conclusions of this paper can be applied for instance to generate a desired

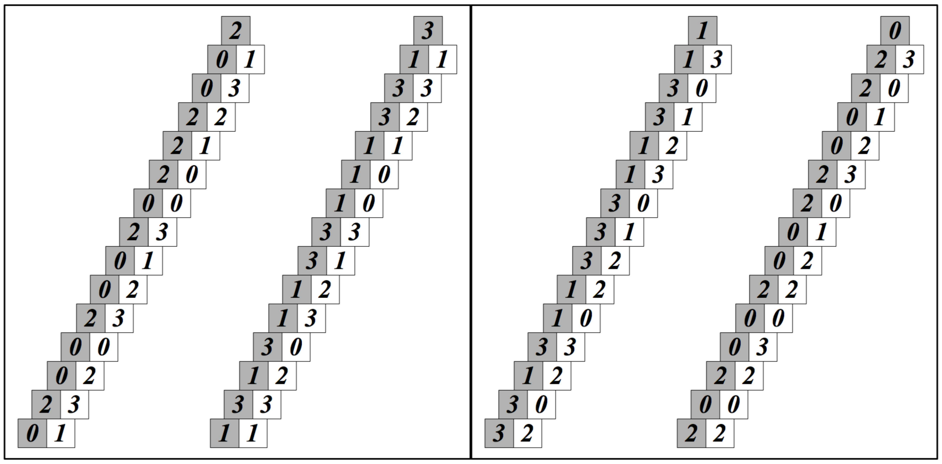

Figure 7: States in the gray cells at the same coordinates are different for each fixed diagonal sequence.

construction during the evolution of the automaton.

The properties exposed in this work can be extended for reversible automata with a neighborhood size greater than 2 according to the simulation explained in Section 3, where the unconventional invertible behavior could be constructed taking blocks of $m_s - 1$ states.

A further work is connecting the combinatorial point of view applied in this document with the algebraic one used by other references in order to achieve a set of deeper properties implemented not only for analyzing reversible automata but applied as well in other topics of symbolic dynamics and graph theory.

References

Adamatzky, A. (ed.) [2003] *Collision-Based Computing* (Springer).

Amoroso, S. & Patt, Y. [1972] "Decision procedures for surjectivity and injectivity of parallel maps for tessellation structures," *J. Comput. Syst. Sci.*, **6**, 448-464.

Boykett, T. [1997] "Comparism of radius 1/2 and radius 1 paradigms in one dimensional reversible cellular automata," http://verdi.algebra.uni-linz.ac.at/˜tim.

Boykett, T. [2003] "Orderly algorithm to enumerate central grupoids," http://www.algebra.uni-linz.ac.at/˜tim/pubs.html.

Boykett, T. [2004] "Efficient exhaustive enumeration of reversible one dimensional cellular automata," *Theor. Comput. Sci.*, **325**, 215-247.

Boykett, T. [2006] "Towards a noetherlike conservation law theorem for one dimensional reversible cellular automata," http://www.algebra.uni-linz.ac.at/˜tim/pubs.html.

Brown, D. B. [1987] "Competition of cellular automata rules," *Complex Syst.*, **1**, 169-180.

Chopard, B. & Droz, M. (eds.) [1998] *Cellular Automata Modeling of Physical Systems* (Cambridge University Press).

Hedlund, G. A. [1969] "Endomorphisms and automorphisms of the shift dynamical system," *Math. Syst. Theory*, **3**, 320-375.

Hillman, D. [1991] "The structure of reversible one-dimensional cellular automata," *Physica D*, **52**, 277-292.

Hillman, D. [2004] "Combinatorial dynamics," http://www.wolframscience.com/ conference/2004/presentations.

Kari, J. [1992a] "Cryptosystems based on reversible cellular automata," http:// citeseer.ist.psu.edu/kari92cryptosystems.html.

Kari, J. [1992b] "On the inverse neighborhoods of reversible cellular automata," in *Lin-*

denmayer Systems. Impacts on theoretical computer science, computer graphics, and developmental biology, eds. Rozenberg, G. & Salomaa, A. (Springer-Verlag) pp. 477-495.

McIntosh, H. V. [1990] "Wolfram's class iv automata and good life," *Physica D*, **45**,105-121.

Moore, E. F. [1970] *Machine models of self-reproduction* (University of Illinois Press).

Moraal, H. [2000] "Graph-theoretical characterization of invertible cellular automata," *Physica D*, **141**,1-18.

Myhill, J. [1963] "The converse of Moore's Garden-of-Eden theorem," *P. Am. Math. Soc.*, **14**, 685-686.

Nasu, M. [1978] "Local maps inducing surjective global maps of one dimensional tesellation automata," *Math. Syst. Theory*, **11**, 327-351.

Seck-Tuoh, J. C. [2002] "Matrix methods and local properties of reversible one-dimensional cellular automata," *J. Phys. A-Math. Gen.*, **35**, 5563-5573.

Seck-Tuoh, J. C. & Chapa, S. V. & Juárez-Martínez, G. & McIntosh, H. V. [2005] "Procedures for calculating reversible one-dimensional cellular automata," *Physica D*, **202**,134-141.

Seck-Tuoh, J. C. & González, M. & Juárez-Martínez, G. & Chapa, S. V. [2003] "Extensions in reversible one-dimensional cellular automata are equivalent with the full shift," *Int. J. of Mod. Phys. C*, **14**(8),1143-1160.

Seck-Tuoh, J. C. & Juárez-Martínez, G. & McIntosh, H. V. [2004] "Correspondence between the temporal and spacial behavior in reversible one-dimensional cellular automata equivalent with the full shift," *TUCS Gen. Publ.*, **32**, 54-63.

Seck-Tuoh, J. C. & Juárez-Martínez, G. & McIntosh, H. V. [2006] "The inverse behavior of a reversible one-dimensional cellular automaton obtained by a single welch diagram," *J. Cell. Automata*, **1**, 25-39.

Toffoli, T. & Margolus, N. [1987] *Cellular Automata Machines* (MIT Press, London).

Toffoli, T. & Margolus, N. [1991] "Invertible cellular automata: a review," in *Cellular Automata: Theory and Experiment,* ed. Gutowitz, H. A. (MIT/North-Holland) pp. 229-253.

von Neumann, J. [1966] *Theory of Self-Reproducing Automata*, ed. Burks, A. W. (University of Illinois Press, Urbana and London).

Wolfram, S. (ed.) [1986] *Theory and Applications of Cellular Automata* (World Scientific, Singapore).