Dynamical aspects in reversible one dimensional cellular automata

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September, 2000

Abstract

We shall analyze the dynamical properties that we can find in reversible one dimensional cellular automata. We take the configuration set of cellular automata as a topological space based on cylinder sets and mappings among them. We will take one dimensional cellular automata with neighborhood radius size 2 for representing the whole set of reversible one dimensional cellular automata. Using also the characterization of reversible cellular automata with block permutations, we will expose some matricial methods for detecting the existence of fixed and periodic points; topologically transitive points; topologically ergodic sets, mixing sets and non-wandering sets. Finally, based on periodic behavior, we will be able to classify dynamical behavior of reversible one dimensional cellular automata.

Keywords: Reversible one dimensional cellular automata, cylinder set, block permutations, dynamical systems.

Contents

1 Introduction

One dimensional cellular automata are discrete dynamical systems characterized by simple interaction of their parts, but at the same time, these systems are able to produce very complex global behavior. The cellular automata theory has three important periods; the concept rises with the work developed by John von Neumann in the middle of the 50's [vN66], he used these systems to demonstrate the possibility of constructing self-reproducing systems. Later, in the beginning of the 70's, we have the work developed by John H. Conway [Gar70], standing out his cellular automata in two dimensions called "Life" which has been widely studied due to the simplicity of its behavior but its capacity to produce complex global behavior. Finally, in the middle of the 80's, Stephen Wolfram [Wol86] was the first in analyze the behavior of the whole set of one dimensional cellular automata with 2 states and neighborhood radius equal 1.

Another important theory in relationship with cellular automata is the study in dynamical systems. The work developed by Henry Poncairé and George Birkhoff among others, had a very strong influence in the work developed by Gustav A. Hedlund [Hed69] at the ends of the 60's. In this paper, Hedlund analyzes in some way the dynamical behavior of cellular automata. In this sense, one result in the work of Wolfram [Wol86] is give the following classification of cellular automata according to their dynamical behavior:

- Class I.- Cellular automata which tend to a fixed point.
- Class II.- Cellular automata which tend to recurrent points.
- Class III.- Cellular automata with chaotic behavior.
- Class IV.- Cellular automata with complex behavior.

However, this classification has the failure that for a given cellular automaton, we can't know which class belongs to, until we observe its behavior. This is a strong problem in the study of dynamical comportment of one dimensional cellular automata since there is not a general characterization of their behavior. Nevertheless, in the case of reversible one dimensional cellular automata we have such a characterization due to the work developed by Jarkko Kari [Kar96]. He explains that the behavior of reversible one dimensional cellular automata is given by block permutations and shifts.

Taking advantage of this characterization, we shall analyze the dynamical behavior in reversible one dimensional cellular automata.

The work has the following organization, section 2 gives the basic concepts about one dimensional cellular automata, explains that every one dimensional cellular automaton can be simulated by another one dimensional cellular automaton with neighborhood size equal 2 and introduces the cylinder sets for handling the configuration space of cellular automata. Section 3 explains the behavior of reversible one dimensional cellular automata using block permutations and shifts, this is done for cellular automata with neighborhood size equal 2 because the other cases can be simulated with this kind of automata. Section 4 gives the basic concepts in dynamical systems that we use in this study. Section 5 shows the dynamical behavior of reversible one dimensional cellular automata using the cylinder sets. We will see that we can use block permutations to give matrix methods that detect the existence of fixed, periodic and transitive points, and ergodic, mixing and non-wandering cylinder sets. Section 6 takes the matrix procedure for detecting periodic points and depending on the equivalence classes that they form with this procedure we can classify the dynamical behavior of reversible one dimensional cellular automata . Section 7 gives some examples of the functionality of this methods in $(4, 1/2)$ reversible one dimensional cellular automata and section 8 has conclusions of this work. Trying to do more clear the exposition and the understanding of this work, section 9 presents a list of symbols commonly used in the development of this paper, making easier their reference.

2 One dimensional cellular automata concepts

In this section we shall view three important topics, the basic concepts of one dimensional cellular automata, the possibility of simulating any one dimensional cellular automaton with another cellular automaton with neighborhood size equal 2, and the specification of cylinder sets.

2.1 One dimensional cellular automata

A one dimensional cellular automaton is formed by a set K of *states*, a sequence c of *cells*, where every cell takes one value of the set K of states. The cardinality of the set K is represented by k, and the sequence c will be called a *configuration* of the automaton. The set of all the possible configurations is called the configuration set C. If the configurations are infinite in both sides, then we can index every cell with an element of the set $\mathbb{Z}$ of integers.

For $n \in \mathbb{Z}^+$, notation K^n represents the set of sequences formed with states of the set K with length n. In this way, notation K^* represents the set of all the finite sequences of states belonging to the set K.

We shall analyze the dynamical behavior given by mappings among configurations of the set C, that is, the kind of trajectories that are formed passing from a given configuration to another applying a given mapping. However, we don't want to analyze all kind of mappings among configurations but only those mappings produced by a local mapping among the cells in a configuration. Given a cell in a configuration c, this cell evolves to a new cell depending on its current state and the state of its r neighbors at each side. Thus, r defines a *neighborhood radius*, and each cell and its r neighbors in both sides form a *neighborhood* with $2r + 1$ cells. We will use the notation (k, r) for representing a one dimensional cellular automata with k states and neighborhood radius r.

Each neighborhood form a new cell and this process is repeated over all the cells in a configuration. The mapping $\varphi : K^{2r+1} \to K$ that exists from neighborhoods to states is named an *evolution rule* of a one dimensional cellular automaton.

The evolution rule φ acts over all the cells in a configuration c forming a new configuration c' that belongs to the configuration set C. Thus, the evolution rule φ induces a global mapping Φ among the configurations of the configuration set C. A special kind of one dimensional cellular automata are those where we can return to previous stages of the system, in this case the evolution rule φ has an inverse rule φ^{-1} that induces a global mapping Φ^{-1} inverse to the global mapping Φ.

This kind of one dimensional cellular automata are called *reversible*, since for any configuration c in the configuration set C, these automata hold that:

$$\Phi \circ \Phi^{-1}(c) = \Phi^{-1} \circ \Phi(c) = c \tag{1}$$

Now we will see that every one dimensional cellular automaton can be simulated by another automaton with neighborhood of length 2.

2.2 Simulating a one dimensional cellular automaton with another of neighborhood size 2

The process of making this simulation is very simple. In a (k, r) one dimensional cellular automaton with evolution rule φ, every neighborhood in the set K^{2r+1} forms a new cell applying the evolution rule φ. In other words, to get a new cell we need $2r + 1$ cells, that is, in general the ancestor sequence has $2r$ cells more than the successor sequence.

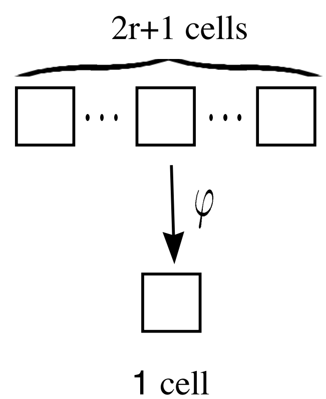


Figure 1: The ancestor of any cell has $2r$ more cells

In this way, if we take a sequence with $2r$ cells, an ancestor of this sequence will have $2r$ cells more, or a total of $4r$ cells. Then, for a sequence with length $2r$ we can make a partition of every ancestor in 2 sequences that don't overlap, each one of them with length of $2r$ cells.

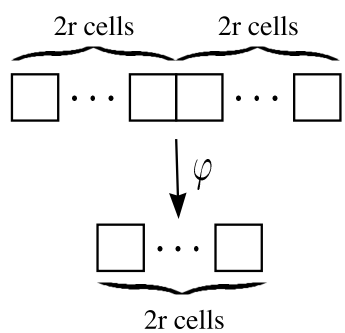


Figure 2: Ancestor of a sequence of $2r$ cells, this ancestor can be divided in 2 sequences of $2r$ cells each one

So, we can define a new alphabet K_i where every sequence of length $2r$ shall be represented by a unique state of the set K_i. Thus, the cardinality of the set K_i is the same that the cardinality of the set K^{2r}. Using this new set K_i of states, we can get a new evolution rule φ_n for simulating the original evolution rule φ; in this case the evolution rule φ_n is a mapping from the set K_i^2 to the set K_i.

With this process a (k, r) one dimensional cellular automaton can be simulated with another $(k_i, 1/2)$ one dimensional cellular automaton, where the cardinality of the set K_i of states is the same that the cardinality of the set K^{2r}.

Of course, in this process we have a considerable increase in the number of states, this is more notorious if the number of states or the neighborhood size of the original one dimensional cellular automaton is big. Nevertheless, if we can characterize a particular property in the behavior of $(k, 1/2)$ one dimensional cellular automata, then we have this characterization for all kind of one dimensional cellular automata.

We are interested in the analysis of the dynamical behavior produced by global mappings in $(k, 1/2)$ reversible one dimensional cellular automata. For this reason, we need one way to define a notion of distance and closeness among the configurations of the configuration set C, we will get this using the cylinder sets.

2.3 Cylinder sets

Since we want to study sequences of states and mappings among these sequences, we need one way for analyzing such sequences. We want a way of organizing the sequences of states such that similar sequences have the same dynamical behavior. For this reason we put special attention in the handling of finite sequences of symbols and the indexing of such sequences in the configurations of the configuration set C.

To place the configurations in the configuration set C in sets with similar configurations both for constructions and for their dynamical behavior, we will use the cylinder sets.

Definition 1. *A cylinder set is a set of configurations such that every configuration ha the same finite sequence of states placed in the same coordinates that the rest of the configurations in the set.*

Following definition 1, given a sequence $w \in K^*$ with length $|\, w \,|$ odd, we have the following definition:

Definition 2. *A centered cylinder set* $\mathcal{C}_{[w]}$ *represents the set of configurations such that their central coordinates are defined by the same sequence* w *of states, that is:*

$$\mathcal{C}_{[w]} = \left\{ c \mid c \in C, c_{[-|w|/2, |w|/2]} = w \right\} \tag{2}$$

Definition 2 means that in a centered cylinder set it only matters the central sequence of states, doesn't matter the states of the rest of the cells.

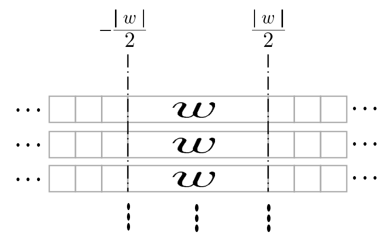


Figure 3: Centered cylinder set specified by the finite sequence w of states

The family $\mathfrak{C}$ is the family of all the possible centered cylinder sets that we get with the configurations of the configuration set C. Using the family $\mathfrak{C}$ of centered cylinder sets we get a topology of the configuration set C such that this is a compact, Hausdorff, and metric topology [Hed69], [Kar96]. Thus, the configuration set C can be taken as an space, this space is represented by the notation $(C, \mathfrak{C})$.

An important feature of the family $\mathfrak{C}$ of centered cylinder sets is that every centered cylinder set is disjoint to the others. For finite sequences w_1 and w_2 elements of the set K^*, if w_1 contains w_2 as a central subsequence, then the intersection of the centered cylinder set $\mathcal{C}_{[w_1]}$ with the centered cylinder set $\mathcal{C}_{[w_2]}$ preserves the centered cylinder set which has more specified coordinates, that is:

$$\mathcal{C}_{[w_1]} \cap \mathcal{C}_{[w_2]} = \mathcal{C}_{[w_1]} \tag{3}$$

But, in the case that w_2 is not a centered subsequence of w_1, then merely the centered cylinder sets specified by these sequences doesn't agree in their central parts, therefore:

$$\mathcal{C}_{[w_1]} \cap \mathcal{C}_{[w_2]} = \varnothing \tag{4}$$

Taking advantage that the configuration space $(C, \mathfrak{C})$ is compact, we can always find a finite covering of this space. In particular, for $n \in \mathbb{Z}^+$, we can take every sequence of states of the set $K^n \subset K^*$ and form the family of centered cylinder sets specified by these sequences:

$$\mathfrak{C}_{K^n} = \left\{ \mathcal{C}_{[w]} \mid w \in K^n \right\} \tag{5}$$

In this way, the family $\mathfrak{C}_{K^n}$ of centered cylinder sets has a finite number of cylinder sets, in particular it has k^n cylinder sets. Since every configuration belongs to a centered cylinder set in the family $\mathfrak{C}_{K^n}$, then we have that:

$$\mathfrak{C}_{K^n} = C \tag{6}$$

So, the family $\mathfrak{C}_{K^n}$ covers the whole configuration space $(C, \mathfrak{C})$ with a finite number of subsets, or centered cylinder sets. Since we can consider every centered cylinder set as a set of nearby configurations, and different centered cylinder sets have more distant configurations, is desirable to specify a function of distance that give us a precise measure of the closeness among configurations in the configuration space $(C, \mathfrak{C})$.

For c and c', configurations of the configuration space $(C, \mathfrak{C})$, we use the distance:

$$d(c, c') = \begin{cases} 0 & if \quad c = c' \\ \frac{1}{1+k} & if \quad c \neq c' \end{cases} \tag{7}$$

where $k \in \mathbb{N}$ and k is the minimum absolute value of the coodinate where $c_{[-k]} \neq c'_{[-k]}$ or $c_{[k]} \neq c'_{[k]}$.

Thus, we have a numerical way for knowing the distance among two configurations in the configuration space $(C, \mathfrak{C})$, in particular, every centered cylinder set $\mathcal{C}_{[w]}$ is a set of configurations where the maximum distance among its elements is defined by:

$$d = \frac{1}{1+k} \quad where \quad k = \frac{\mid w \mid}{2} + 1 \tag{8}$$

Using the centered cylinder sets in $(k, 1/2)$ reversible one dimensional cellular automata, we shall analyze the dynamical properties of the global mapping Φ induced by an invertible evolution rule φ. But first, we will explain the characterization of the behavior of these systems.

3 Characterization of $(k, 1/2)$ reversible one dimensional cellular automata

A characterization of reversible one dimensional cellular automata is possible if we use block permutations and shifts, this process will be useful for analyzing dynamical behavior of such systems.

The main result in the work of Kari [Kar96] is that the action of every reversible one dimensional cellular automaton can be represented by the process of applying 2 block permutations and a shift among them. To explain the previous affirmation, first we will explain which are the properties of these reversible systems.

3.1 Properties of reversible one dimensional cellular automata

Based on the work developed by Hedlund [Hed69], we can notice the following:

Proposition 1. *Reversible one dimensional cellular automata have the following properties:*

1. *Every finite sequence of states in the set K^* have k^{2r} ancestors or finite sequences that generate it employing the evolution rule φ*
2. *The ancestors of every finite sequence in the set K^* have L different left sequences,* 1 *unique central part and R different right sequences, holding that $LR = k^{2r}$*

The first statement in Proposition 1 can be called the principle of uniform multiplicity of ancestors [McI91b]; and the values of L and R in the second statement are defined by Hedlund as the Welch indices [Hed69]. In this way, a reversible one dimensional cellular automaton holds that every sequence has the same number of ancestors that all the other sequences, and the ancestors of each sequence share a common central part , leaving the differences in the extremes.

3.2 Block permutations

Using the properties of reversible one dimensional cellular automata we can characterize the evolution of these systems. For a reversible one dimensional cellular automaton with invertible rules φ and φ^{-1}, take the greatest value of the neighborhood radius among both evolution rules, and represent these rules with this neighborhood radius, in such a way that both rules have the same neighborhood size.

Since both rules have the same neighborhood size, then a sequence of $2r+1$ cells maps to a single state applying the inverse evolution rule φ^{-1}, but the same sequence has k^{2r} ancestors with the evolution rule φ. In this way we have that a sequence of $2r+1$ has k^{2r} ancestors which share a unique common central cell, as is presented in Figure 4 .

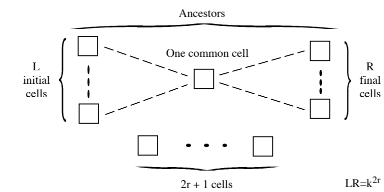


Figure 4: Ancestors of a neighborhood in a reversible one dimensional cellular automaton

This is extensible for sequences of states with a length greater than $2r+1$, a long sequence can be taken as successive overlaps of sequences with length $2r+1$. Therefore we have that for $n \geq 2r+1$, a sequence of length n has k^{2r} ancestors, each one of them with $n+2r$ cells, where the ancestors share a common sequence with length $n-2r$; this is showed in Figure 5.

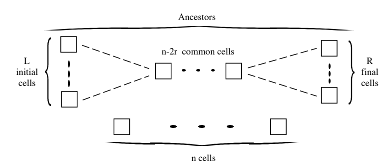


Figure 5: Ancestors of a sequence of n cells in a reversible one dimensional cellular automaton

Take a sequence of $4r$ cells, this sequence has k^{2r} ancestors of length $6r$ cells, and these ancestors have a common central sequence with $2r$ cells, as we can see in Figure 6.

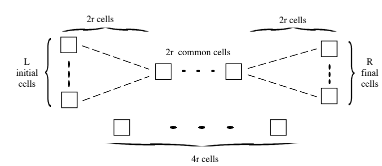


Figure 6: Ancestors of a sequence of $4r$ cells in a reversible one dimensional cellular automaton

The same behavior exists for the inverse evolution rule φ^{-1} but in inverse direction and with inverted values of its Welch indices. That is, the index L in the inverse evolution rule φ^{-1} has the same value that the index R in the original evolution rule φ, and the index R in φ^{-1} has the same value that the index L in φ. With the construction in Figure 6, we can define 2 sets L_{φ} and R_{φ}, where the elements of L_{φ} are sequences of length $2r$ cells and the left ancestor sequences of each one of them, also of length $2r$ cells. This is analogous for constructing the elements of set R_{φ}.

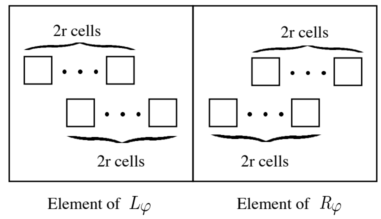


Figure 7: Elements of the sets L_φ and R_φ

Thus, the set L_φ has as many elements as $| L_\varphi | = Lk^{2r}$ and the set R_φ has as many elements as $| R_\varphi | = Rk^{2r}$. We now define two sets, X and Y, such that the cardinality of X is $| X | = | L_\varphi |$ and the cardinality of the set Y is $| Y | = | R_\varphi |$. Then, there exists a bijection both from the set L_φ to the set X, and from the set R_φ to the set Y. In this way, we can define two block permutations p_1 and p_2.

The permutation p_1 goes from the set of sequences with length $6r$ cells to all the possible sequences with the form $x_i y_j$, where $x_i \in X$ and $y_j \in Y$, for $0 \leq i \leq Lk^{2r}$ and $0 \leq j \leq Rk^{2r}$. The second permutation p_2 is almost analogous, it goes from the set of sequences with length $6r$ cells to the set of all the possible sequences with the form $y_j x_i$. With these permutations, we can represent the evolution of a reversible one dimensional cellular automaton as the composition $p_1 \circ p_2^{-1}$ of two block permutations and a shift of length $3r$ cells between both permutations.

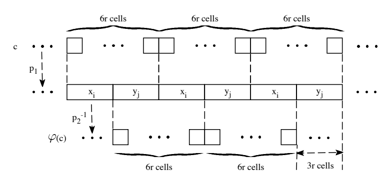


Figure 8: Evolution of a reversible one dimensional cellular automaton represented by the composition $p_1 \circ p_2$ of block permutations and a shift of $3r$ cells among the permutations

We shall use these block permutations in $(k, 1/2)$ reversible one dimensional cellular automata to analyze the dynamical behavior of these systems. The following section establishes the basic concepts of dynamical system theory that we use in this study.

4 Dynamical system concepts

The idea behind dynamical systems theory is studying, understanding, and estimating the long place behavior of a system that changes in time. The characterization of this behavior consists in knowing which are the conditions of a system such that it has a particular comportment; some examples of these comportments are the following:

- The system has a periodic behavior
- The system recurrently returns to a given set
- The system goes to all the possible sets that cover the space of the system
- The system never leaves a given set

Using some useful topological concepts we define a dynamical systems in the following way:

Definition 3. *A dynamical system (X, Ψ) consists of a metric, compact space X and a continuous mapping $\Psi : X \to X$ that maps elements of the space X to the space X.*

A consequence of definition 3 is the concept of an orbit of a given point in the space X:

Definition 4. *In a dynamical system (X, Ψ), an orbit is the trajectory that a given point $x \in X$ has in the space X with the successive application of the mapping Ψ over the point x*

The reason for the compactness of the space X is that such spaces can be covered and thereby represented by a finite number of sets. With this, the orbit of a point x in the space X can be described by the finite number of sets that it reaches, and this feature provides an easier analysis. In the study of the dynamical behavior in $(k, 1/2)$ reversible one dimensional cellular automata, we are interested in characterizing the orbits of the configurations c in the configuration space $(C, \mathfrak{C})$ applying the global mapping Φ^{-1}. In particular, we want to describe the periodic, recurrent and the transitive behavior, for this motive, based on the works of J. de Vries [dV93] and Clark Robinson [Rob95] we present some definitions of these kinds of behavior.

Definition 5. *A point x in a dynamical system (X, Ψ) is called a periodic point with minimum period n if $\Psi^n(x) = x$ and $\Psi^j(x) \neq x$ for $0 \leq j \leq n$*

Definition 5 says that after n iterations of the mapping Ψ, the point x comes back to the same place. If a point x in a dynamical system (X, Ψ) has period equal 1, then it is a *fixed* point. The analysis of periodic points and periodic orbits is the beginning in the study of a dynamical system. Now, we can ask if there are orbits that come back not to the same point but to the same open set of the initial point. If this happens for every open set in the space X, then we have the following definition:

Definition 6. *A dynamical system (X, Ψ) is non-wandering if for every open set $\mathcal{O}$ of the space X there exists an integer $n > 0$ such that $\Psi^n(\mathcal{O}) \cap \mathcal{O} \neq \varnothing$, that is, there exists a point $x \in \mathcal{O}$ such that $\Psi^n(x) \in \mathcal{O}$*

In the previous definition, we are using the topologycal nature of the space X utilizing the open sets of this space for characterizing the non-wandering orbits that mapping Ψ can generate. Until now, we have defined orbits with a recurrent behavior, but another question is if the dynamical behavior of the system is such that an orbit can reach each one of the neighborhoods that cover the space X. This idea gives the following definition:

Definition 7. *A dynamical system (X, Ψ) is topologically transitive if there exists a point x in the space X such that for all open set $\mathcal{O}$ in the space X there is an integer n such that $\Psi^n(x) \cap \mathcal{O} \neq \varnothing$*

Definition 7 establishes a point that cross all the neghbborhoods in a finite covering of a dynamical system (X, Ψ). Another transitive behaviors are also possible:

Definition 8. *A dynamical system (X, Ψ) is topologically ergodic if for all pair $\mathcal{O}_1$ and $\mathcal{O}_2$ of open sets of the space X, we have that $\Psi^n(\mathcal{O}_1) \cap \mathcal{O}_2 \neq \varnothing$*

This definition looking for the existence of a topologically transitive point in every open set of the space X. Periodicity and transitivity of points and open sets are the dynamical behaviors that we shall analyse in $(k, 1/2)$ reversible one dimensional cellular automata. In addition, we will define one more concept, if there exists an orbit such that it reaches a given open set and it remains there forever.

Definition 9. *A dynamical system (X, Ψ) is topologically mixing if for all pair $\mathcal{O}_1$ and $\mathcal{O}_2$ of open sets in the space X, there exists an integer n_0 such that $\Psi^n(\mathcal{O}_1) \cap \mathcal{O}_2 \neq \varnothing$ for all $n \geq n_0$*

In section 5 we will develop some matrix methods for detecting the existence of the different points and sets described in the previous definitions in $(k, 1/2)$ reversible one dimensional cellular automata.

5 Dynamical behavior of $(k, 1/2)$ reversible one dimensional cellular automata

In one dimensional cellular automata, we will consider the global mapping Φ among configurations of the configuration set C induced by an evolution rule φ as the mapping generating the dynamical behavior in the configuration space $(C, \mathfrak{C})$.

The orbit described by any configuration is the progressive evolution produced by the iteration of the global mapping Φ over the same configuration, and the regions that this orbit visits are the cylinder sets covering the configuration set C.

Right now a complete characterization of one dimensional cellular automata doesn't exists. For this reason is very difficult to know which are the conditions for classifying the evolution of a one dimensional cellular automaton, doesn't matter the simplicity of its evolution rule. However, for the reversible case we have this characterization as we see in section 3. This result will be used to define some matrix methods for detecting the kind of orbits described in section 4.

We only analyse the case of $(k, 1/2)$ reversible one dimensional cellular automata because all the cases can be simulated whit this kind of automata.

5.1 Dynamical behavior

Our objective in studying dynamical behavior of reversible one dimensional cellular automata is understanding the conditions that an initial configuration must hold to evolves in a particular behavior. In this way, in one dimensional cellular automata we define an orbit in the following way:

Definition 10. *In one dimensional cellular automata, for $i \in \mathbb{N}$ and configurations $c_i \in C$, an orbit $e = \{c_0, c_1, \ldots, c_i, \ldots\}$ is the sequence of configurations such that configuration c_{i+1} is the evolution of the configuration c_i*

Given an orbit e, its behavior is characterized by the cylinder sets that it reaches and in what way it passes these cylinder sets. For covering the configuration space $(C, \mathfrak{C})$, we only use the set K^3 of finite sequences with 3 cells. We do this because for $(k, 1/2)$ reversible one dimensional cellular automata, sequences of 3 cells are that we need to form the block permutations, that is, sequences of length $6r$. Block permutations define the transition from one sequence of 3 cells to another sequence with the same length, so we can see that as the transition from one cylinder set to another cylinder set. There exists a shift among these sequences of length $3r$, or of length $3/2$ cells with neighborhood size 2. To obtain a shift of equal length that the sequences, we define an evolution rule $\varphi' = \varphi \circ \varphi$, that is, an evolution rule that is the composition of the original evolution rule. This process is not necessary but is useful because allows us to work with centered cylinder sets.

To have a simpler notation in this section, we shall use the symbol φ to reference the composition of the original evolution rule, and the symbol φ^{-1} to reference the composition of the inverse evolution rule. These invertible evolution rules induce global mappings Φ and Φ^{-1} that map one sequence of 3 cells to another sequence with the same length and a shift among them of 3 cells. In other words, for an orbit e defined with this evolution rule, we have a mapping from a centered cylinder set $\mathcal{C}_{[c_{i[-1,1]}]}$ to a centered cylinder set $\mathcal{C}_{[c_{i+1[-1,1]}]}$ as we can see in Figure 9.

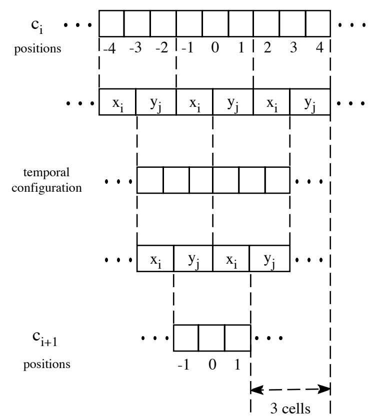


Figure 9: Passing from the centered cylinder set $\mathcal{C}_{[c_{i[-1,1]}]}$ to the centered cylinder set $\mathcal{C}_{[c_{i+1[-1,1]}]}$ using the composition of evolution rules in a $(k, 1/2)$ reversible one dimensional cellular automaton

Based on block permutations, we present simple matrix methods for detecting the existence of different kind of orbits.

5.2 Periodic behavior of $(k, 1/2)$ reversible one dimensional cellular automata

In the paper of Hedlund [Hed69], section 7 is devoted to analyse the dynamical behavior of the shift systems; based on his work we will do an analysis of periodic orbits in $(k, 1/2)$ reversible one dimensional cellular automata.

Suppose that a given configuration c in the configuration set C is formed by the successive repetition of a finite sequence w of n cells. Thus, the states that form this configuration have a period n. Now, suppose that we apply an invertible evolution rule φ, since the action of this rule is a block permutation, we can characterize the periodical behavior under the global mapping Φ induced by φ of a configuration c formed by a periodical finite sequence $w \in K^n$.

Theorem 1. *Given a $(k, 1/2)$ reversible one dimensional cellular automaton and a configuration c formed with the successive repetition of a finite sequence w of length n, the maximum period of the orbit formed with the configuration c is k^{3n}*

Proof. The configuration c is formed with a periodic finite sequence w of n cells, take a sequence of w_1 of $3n$ cells in the configuration c. The sequence w_1 has a period of length $3n$ because is the repetition of a periodic sequence w of n cells. But w_1 also has n sequence of 3 cells each one of them, applying the global mapping Φ, every sequence of 3 cells maps to another unique sequence of 3 cells as is showed by the block permutations. Then the complete sequence w_1 of $3n$ cells maps to an unique sequence w_2 of $3n$ cells too.

Since all the sequences of length $3n$ cells are in the finite set K^{3n}, and the cardinality of the set K^{3n} is $\mid K^{3n} \mid = k^{3n}$, then in some moment during the evolution of the automaton we have to repeat the same sequence w_1. Thus, the maximum period of the configuration c formed with the repetition of a finite sequence w of length n is k^{3n}. □

In the general case of (k, r) reversible one dimensional cellular automata, Theorem 1 defines a maximum period of $6rn$ steps; where r is the neighborhood radius and n is the length of the finite sequence w whose repetition forms the configuration c. We have to point out that this maximum period in most cases is a bad quote, because the practical experience shows that this period is much smaller.

Periodic orbits e of period n goes from a centered cylinder set to the same cilinder set. Since every sequence w of length 3 cells defines a centered cylinder set, then the family of all centered cilinder sets forms a finite covering of the configuration space $(C, \mathfrak{C})$. Then, a consequence of Theorem 1 and using Definition 6 is the following:

Corollary 1. *For every sequence $w \in K^3$, the centered cylinder set $\mathcal{C}_{[w]}$ is a non-wandering set.*

Proof. Take every sequence w in K^3, and form a configuration c with the successive repetition of w. Then the configuration c belongs to the centered cylinder set $\mathcal{C}_{[w]}$ and is periodic with finite period by Theorem 1. Thus, the orbit of c returns to the same centered cylinder set $\mathcal{C}_{[w]}$ and therefore is non-wandering □

Now we will use block permutations for detecting the periodical behavior of these systems.

5.3 Detecting some periodical behavior in $(k, 1/2)$ reversible one dimensional cellular automata

We can use transitions among block permutations to find periodical orbits in $(k, 1/2)$ reversible one dimensional cellular automata. Take the set K^3 formed with all the sequences of 3 cells, with these sequences we can form k^3 configurations, each one of them obtained with the succesive repetition of one sequence in the set K^3. In this way, for a configuration of this kind, we can know which is its succesor configuraton using the block permutations.

For $0 \leq i \leq k^3$, every configuration c_i has the form $\ldots w_i w_i w_i \ldots$ for $w_i \in K^3$ and every w_i maps to an unique block with the form $x_i y_i$ for $x_i \in X$ and $y_i \in Y$, then the whole configuration c_i maps to a sequence of blocks with the form $\cdots x_i y_i x_i y_i x_i y_i \cdots$. Thus, we have only two kind of blocks, $x_i y_i$ and $y_i x_i$, and applying the permutation p_2^{-1}, the block $y_i x_i$ maps to an unique sequence $w_j \in K^3$. Making twice this process we have a mapping from the block $x_i y_i$ to another block $x_k y_k$ placed in the same coordinates in relation to the positions in the configuration c_i, this is showed in Figure 10.

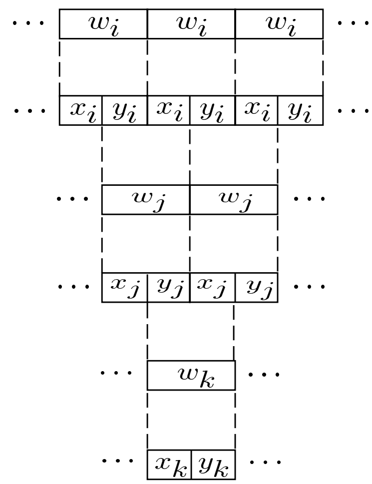


Figure 10: Mapping from a block $x_i y_i$ to a block $x_k y_k$ placed in the same position

Applying the previous process to all the k^3 configurations, then we have a bijective mapping among the blocks xy because the cellular automaton is reversible, so every block has one ancestor and successor. With this, we can form a connectivity relation whose indices are blocks with the form xy and the elements show the mapping from one block to another.

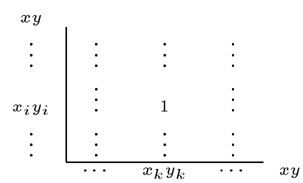


Table 1: Connectivity relation for periodic behavior defined with blocks xy

We can define an equivalence relation in the connectivity relation defined in Table 1. If the block $x_i y_i$ maps to $x_k y_k$ and this block maps to $x_m y_m$, then there is a mapping from $x_i y_i$ to $x_m y_m$. We can do the transitive closure of the connectivity relation (for example, using the Warshall algorithm), and get the equivalence relation. Due to the periodical behavior of $(k, 1/2)$ reversible one dimensional cellular automata, every block xy returns to itself and the transitive closure is also reflexive. If there exists a mapping from the block $x_i y_i$ to the block $x_m y_m$, then due also to the periodical behavior, there exists a mapping from $x_m y_m$ to $x_i y_i$, thus the relation is symmetric and we have an equivalence relation, as is presented in Figure 11. Every class in this equivalence relation represents a set of periodic configurations. The period of every class is the number of elements that it has.

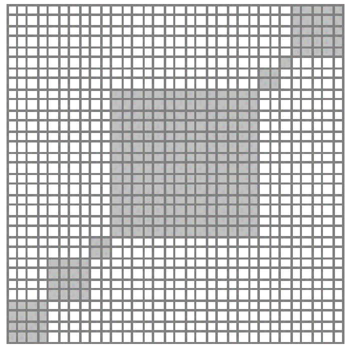


Figure 11: Representation of some equivalence relation defined by the transitive closure of the connectivity relation described in Table 1

5.4 Transitive behavior of $(k, 1/2)$ reversible one dimensional cellular automata

We have seen that some restrictions in the configurations define both periodic orbits and non-wandering centered cylinder sets in $(k, 1/2)$ reversible one dimensional cellular automata. Now is desirable to consider not only one but all the possible mappings that a sequence $w \in K^3$ has to. This can be done using again block permutations and the process is the following:

1. For a sequence w_i of states in the set K^3, take its mapping to an unique block $x_i y_i$ using the block permutation p_1.
2. Associate the element x_i with all the elements y in the set Y and the element y_i with all the elements x in the set X.
3. With the associations yx_i and $y_i x$ and using the block permutation p_2^{-1}, form the respective list of the mappings from these associations to sequences w_j and w_k of states in the set K^3.
4. Using the block permutation p_1, every list of sequences w_j and w_k of states defined by yx_i and $y_i x$, maps respectively to a list of blocks with the form $x_j y_j$ and $x_k y_k$.
5. From the last lists, take only all the different elements y_j in the first list and all the different elements x_k in the second list.
6. Form the cartesian product of such lists. This cartesian product form a new list of blocks with the form $y_j x_k$.
7. The list of blocks $y_j x_k$ maps to a list of sequences w_m using the block permutation p_2^{-1}. Thus, we have a subset of the set K^3.

In this way, we have that an initial sequence w_i maps to a set of sequences w_m placed in the same position and therefore we have all the possible mappings from sequences to sequences in the set K^3 as is showed in Figure 12.

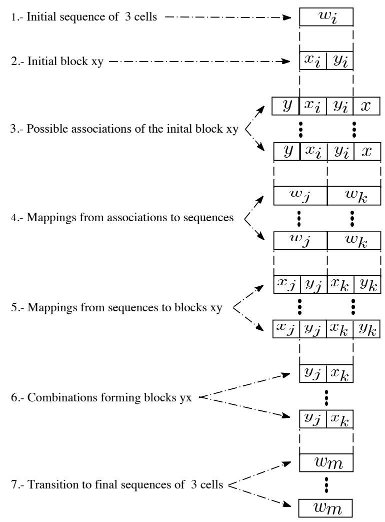


Figure 12: Description of all the possible mappings from a sequence w_i to a set of sequences w_m using block permutations

Using all the sequences w of states in the set K^3 for defining centered cylinder sets $\mathcal{C}_{[w]}$, the previous process defines the possible mappings from a centered cylinder set to other centered cylinder sets. In this way, we can use this process for detecting transitive behavior among centered cylinder sets that cover all the configuration space $(C, \mathfrak{C})$.

5.5 Detecting transitive behavior in $(k, 1/2)$ reversible one dimensional cellular automata

Using the process presented in section 5.4, we can define a transition relation among sequences w in the set K^3. In this way, we have a surjective mapping from the set K^3 to itself, since every sequence has at least one ancestor. In this transition relation, the indices are sequences w of 3 cells and the elements show the mapping from one sequence w_i to a set of sequences w_m as is presented in Table 2.

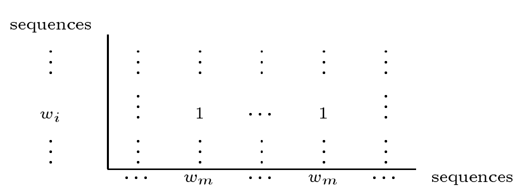


Table 2: Transitive relation among sequences of the set K^3 defined by block permutations

As in the connectivity relation defined in Table 1, we can do the transitive closure of the transition relation. Depending of the number of classes that we get with this process, we will detect different kinds of transitive behavior among centered cylinder sets defined by sequences in the set K^3.

Lemma 1. *If the transitive closure of the transition relation in Table 2 forms one unique class, then the $(k, 1/2)$ reversible one dimensional cellular automaton is topologically ergodic*

Proof. For every sequence w in the set K^3, if the transitive closure forms an unique class, then it shows that there exists an orbit e from the centered cilinder set $\mathcal{C}_{[w]}$ to all the others centered cylinder sets, and therefore the centered cylinder set $\mathcal{C}_{[w]}$ is topologically ergodic. Since we have only one unique class, then all sequences w in the set K^3 carry out with the previous statement, and the centered cylinder sets defined by these sequences are topologically ergodic. □

In the configuration space $(C, \mathfrak{C})$, if a $(k, 1/2)$ reversible one dimensional cellular automaton defines a topologically ergodic dynamical system then for every $w \in K^3$, there exists a configuration c which belongs to the centered cylinder set $\mathcal{C}_{[w]}$ and it has an orbit through all the other centered cylinder sets defined by the sequences in the set K^3. Thus, a consequence of Lemma 1 is the following:

Corollary 2. *A topologically ergodic* $(k, 1/2)$ *reversible one dimensional cellular automaton has topologically transitive configurations.*

Finally, the transition relation and its transitive closure give us important information about the mixing behavior of $(k, 1/2)$ reversible one dimensional cellular automata. In the transition relation, the $1's$ in the principal diagonal indicate centered cylinder sets which can return to the same centered cylinder set in one step. In other words, the principal diagonal shows centered cylinder sets that can be fixed.

Using the principal diagonal and the transitive closure of the transition relation, we have the following result:

Theorem 2. *If a* $(k, 1/2)$ *reversible one dimensional cellular automaton is topologycally ergodic and its transition relation has a non-zero principal diagonal, then the automaton is topologically mixing*

Proof. Because the principal diagonal is non-zero, there exists at least one centered cylinder set that can be fixed. Since the automaton is ergodic, there exists an orbit e from any centered cylinder set to a centered cylinder set that can be fixed. Thus, this orbit in this centered cylinder set can remain there n steps for $n \in \mathbb{Z}^+$, and then goes out to another centered cylinder set. So, beginning in every centered cylinder set, we can reach any other centered cylinder set in the number of steps that we want, therefore every centered cylinder set is topologically mixing. □

In this way, we have defined simple matrix methods that using the properties of block permutations and transitive closures detect periodical and transitive behavior. Of course, a big problem is that these methods only detect the existence of these orbits but they doesn't give an explicit example of them.

6 Classifing $(k, 1/2)$ reversible one dimensional cellular automata using their periodical behavior

The methods presented in section 5 could be useful for comparing the dynamical behavior of $(k, 1/2)$ reversible one dimensional cellular automata and thereby we can get a dynamical classification of these systems.

In particular we will use the transitive closure of the connectivity relation defined in Table 1 for analyzing periodical behavior. Take two $(k, 1/2)$ reversible one dimensional cellular automata and the transitive closures of their connectivity relations, then we consider that these automata belong to the same dynamical class if:

1. The number of equivalence classes is the same in both equivalence relations.
2. There exists an isomorphism from every equivalence class in one equivalence relation to another equivalence class in the other equivalence relation.

An example of equivalence classes that belong to the same dynamical class is showed in Figure 13.

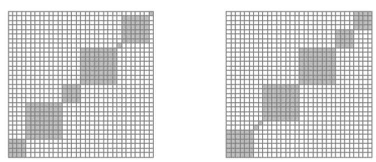


Figure 13: Equivalence relations of the connectivity matrices that belong to the same dynamical class

With this process we are comparing the quantitative behavior of the transitive closure of the connectivity relations, that is, we are contrasting if the block permutations of different $(k, 1/2)$ reversible one dimensional cellular automata have the same periodical behavior.

This way of classifing $(k, 1/2)$ reversible one dimensional cellular automata is based more in an experimental and numerical approach than in a theoretical one, besides, this process is easy for calculating if the $(k, 1/2)$ reversible one dimensional cellular automata have not a big number of states.

7 Examples

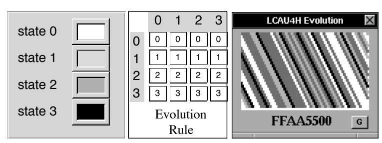
In this section, we present some examples of the matrix methods developed in sections 5 and 6 for detecting and classifying $(4, 1/2)$ reversible one dimensional cellular automata. These methods were implemented in the system RLCAU that calculates $(k, 1/2)$ reversible one dimensional cellular automata using block permutations.

Every cellular automaton in the following examples has an hexadecimal number that identifies each one of these cellular automata. This hexadecimal number is calculated taking the evolution rule, sorting it in descending lexicografical order and dividing this sort in pairs of two neighborhoods, every pair has associated an unique hexadecimal symbol depending of the evolution of its neighborhoods. In this way we have 8 pairs of two neighborhoods, then we have an hexadecimal number of 8 symbols identifying every evolution rule in $(4, 1/2)$ one dimensional cellular automata.

We shall use a matrix for representing the evolution rule of a $(4, 1/2)$ one dimensional cellular automaton. In this matrix, the indices represent partial neighborhoods, thereby the positions of the elements are complete neighborhoods. Every element represents the evolution of every neighborhood. We also use the system NXLCAU [McI90] for developing these examples.

7.1 $(4, 1/2)$ reversible cellular automaton, rule FFAA5500

This automaton has Welch indices $L = 1$ and $R = 4$. The evolution rule, an example of the evolution, and its block permutations are the following:



Block permutations

Sequence				p1		p2		Sequence				p1		p2	
0	0	0	->	0	0	0	0	2	0	0	->	2	0	6	0
0	0	1	->	0	4	0	1	2	0	1	->	2	4	6	1
0	0	2	->	0	8	0	2	2	0	2	->	2	8	6	2
0	0	3	->	0	12	0	3	2	0	3	->	2	12	6	3
0	1	0	->	0	5	4	0	2	1	0	->	2	5	10	0
0	1	1	->	0	1	4	1	2	1	1	->	2	1	10	1
0	1	2	->	0	9	4	2	2	1	2	->	2	9	10	2
0	1	3	->	0	13	4	3	2	1	3	->	2	13	10	3
0	2	0	->	0	6	8	0	2	2	0	->	2	6	2	0
0	2	1	->	0	10	8	1	2	2	1	->	2	10	2	1
0	2	2	->	0	2	8	2	2	2	2	->	2	2	2	2
0	2	3	->	0	14	8	3	2	2	3	->	2	14	2	3
0	3	0	->	0	7	12	0	2	3	0	->	2	7	14	0
0	3	1	->	0	11	12	1	2	3	1	->	2	11	14	1
0	3	2	->	0	15	12	2	2	3	2	->	2	15	14	2
0	3	3	->	0	3	12	3	2	3	3	->	2	3	14	3
1	0	0	->	1	0	5	0	3	0	0	->	3	0	7	0
1	0	1	->	1	4	5	1	3	0	1	->	3	4	7	1
1	0	2	->	1	8	5	2	3	0	2	->	3	8	7	2
1	0	3	->	1	12	5	3	3	0	3	->	3	12	7	3
1	1	0	->	1	5	1	0	3	1	0	->	3	5	11	0
1	1	1	->	1	1	1	1	3	1	1	->	3	1	11	1
1	1	2	->	1	9	1	2	3	1	2	->	3	9	11	2
1	1	3	->	1	13	1	3	3	1	3	->	3	13	11	3
1	2	0	->	1	6	9	0	3	2	0	->	3	6	15	0
1	2	1	->	1	10	9	1	3	2	1	->	3	10	15	1
1	2	2	->	1	2	9	2	3	2	2	->	3	2	15	2
1	2	3	->	1	14	9	3	3	2	3	->	3	14	15	3
1	3	0	->	1	7	13	0	3	3	0	->	3	7	3	0
1	3	1	->	1	11	13	1	3	3	1	->	3	11	3	1
1	3	2	->	1	15	13	2	3	3	2	->	3	15	3	2
1	3	3	->	1	3	13	3	3	3	3	->	3	3	3	3

Figure 14: Evolution of the $(4, 1/2)$ reversible one dimensional cellular automaton rule $FFAA5500$

The connectivity relation associated with this automaton is the following:

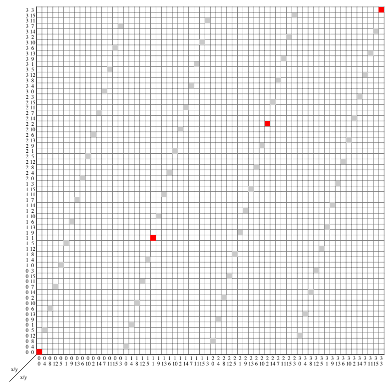


Figure 15: Connectivity relation of the $(4, 1/2)$ reversible one dimensional cellular automaton rule $FFAA5500$, the dark points show fixed configurations

The transitive closure of connectivity relation associated with this automaton is the following:

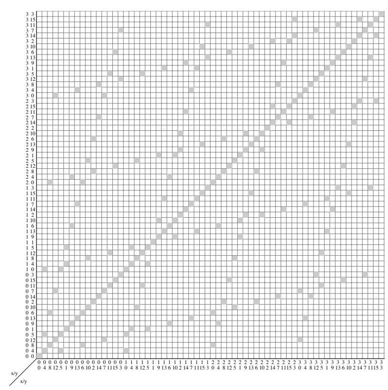


Figure 16: Transitive closure of the connectivity relation of the $(4, 1/2)$ reversible one dimensional cellular automaton rule $FFAA5500$

If we rearrange this transitive closure, we obtain the following classes:

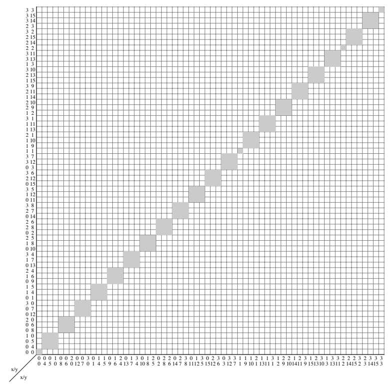


Figure 17: Rearrange of the transitive closure of the connectivity relation of the $(4, 1/2)$ automaton rule $FFAA5500$

In this case we see that the automaton have 24 equivalence classes, 20 of 3 elements each one and 4 of one element. Take the block $1, 8$ representing the sequence of states 102. This block has period 3, so the configuration formed with repetitions of the sequence 102 must have period 3. We have to remember that we are using the composition of the original evolution rule for keeping the same position when we compare configurations. In this way, the period 3 is truly a period 6 in the evolution of the automaton. An example of this periodical behavior is the following:

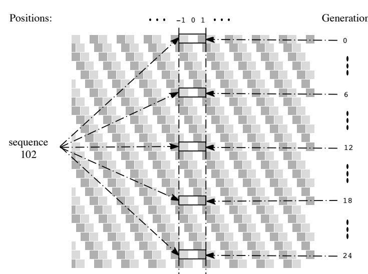


Figure 18: Period 6 corresponding to a period 3 using the composition of the evolution rule in the initial configuration formed with repetitions of the sequence 102

Now, we will see all the possible mappings among sequences of 3 cells using the process described in section 5.4. For example, the mapping of 203 is the following:

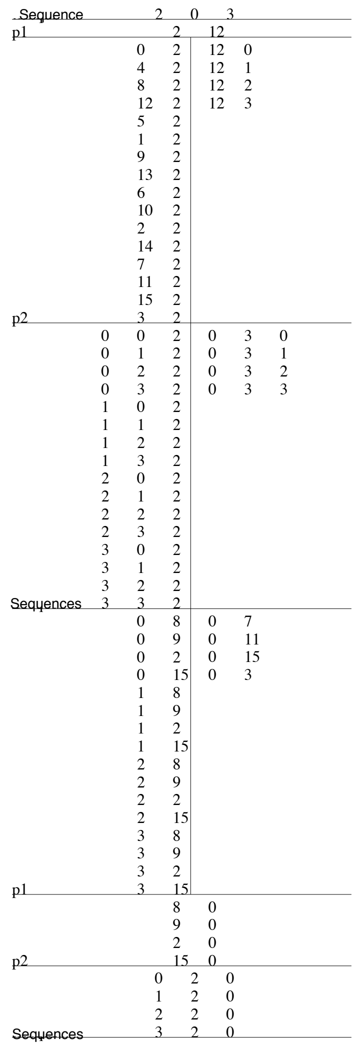


Figure 19: All the possible mappings from the sequence 203

Calculating all the possible mappings among sequences of 3 cells, and taking such sequences as centered cylinder sets, we have the following mapping among centered cylinder sets:

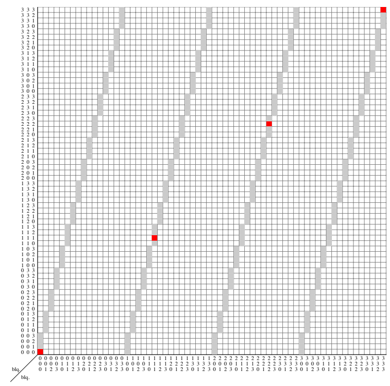


Figure 20: Mapping among centered cylinder sets, the dark points indicate recurrent centered cylinder sets

The transitive closure of the mapping among centered cylinder sets is the following:

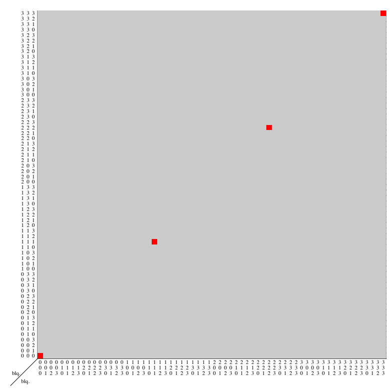


Figure 21: Transitive closure of the mapping among centered cylinder sets, the dark points indicate recurrent centered cylinder sets

Since we only have one equivalence class and there exists centered cylinder sets that can be fixed, then this automaton has topologically mixing orbits. For example, we can form an orbit from the centered cylinder set $\mathcal{C}_{[203]}$ to the centered cylinder set $\mathcal{C}_{[312]}$ in 6 steps, corresponding to 12 evolutions because the composition of the evolution rule. We use the recurrent centered cylinder set $\mathcal{C}_{[222]}$ for constructing such an orbit.

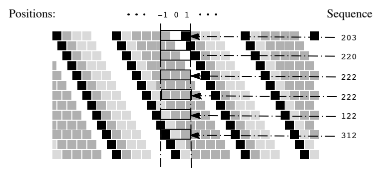


Figure 22: Orbit from the centered cylinder set $\mathcal{C}_{[203]}$ to the centered cylinder set $\mathcal{C}_{[312]}$ in 6 steps

But, since the centered cylinder set $\mathcal{C}_{[222]}$ can be fixed, we can use it to get an orbit from the centered cylinder set $\mathcal{C}_{[203]}$ to the centered cylinder set $\mathcal{C}_{[312]}$ in 7 steps.

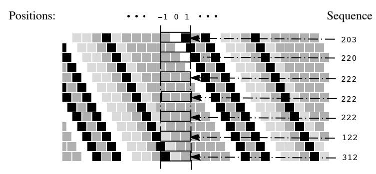
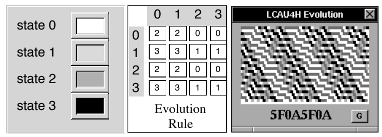


Figure 23: Orbit from the centered cylinder set $\mathcal{C}_{[203]}$ to the centered cylinder set $\mathcal{C}_{[312]}$ in 7 steps

7.2 $(4, 1/2)$ reversible cellular automaton, rule 5F0A5F0A

This automaton has Welch indices $L = 2$ and $R = 2$. The evolution rule, an example of the evolution, and its block permutations are the following:



Block permutations

Sequence				p1		p2	
0	0	0	->	2	2	0	0
0	0	1	->	2	6	0	1
0	0	2	->	2	0	0	6
0	0	3	->	2	4	0	7
0	1	0	->	2	7	4	0
0	1	1	->	2	3	4	1
0	1	2	->	2	5	4	6
0	1	3	->	2	1	4	7
0	2	0	->	4	2	0	4
0	2	1	->	4	6	0	5
0	2	2	->	4	0	0	2
0	2	3	->	4	4	0	3
0	3	0	->	4	7	4	4
0	3	1	->	4	3	4	5
0	3	2	->	4	5	4	2
0	3	3	->	4	1	4	3
1	0	0	->	3	2	5	0
1	0	1	->	3	6	5	1
1	0	2	->	3	0	5	6
1	0	3	->	3	4	5	7
1	1	0	->	3	7	1	0
1	1	1	->	3	3	1	1
1	1	2	->	3	5	1	6
1	1	3	->	3	1	1	7
1	2	0	->	5	2	5	4
1	2	1	->	5	6	5	5
1	2	2	->	5	0	5	2
1	2	3	->	5	4	5	3
1	3	0	->	5	7	1	4
1	3	1	->	5	3	1	5
1	3	2	->	5	5	1	2
1	3	3	->	5	1	1	3

Sequence				p1		p2	
2	0	0	->	6	2	2	0
2	0	1	->	6	6	2	1
2	0	2	->	6	0	2	6
2	0	3	->	6	4	2	7
2	1	0	->	6	7	6	0
2	1	1	->	6	3	6	1
2	1	2	->	6	5	6	6
2	1	3	->	6	1	6	7
2	2	0	->	0	2	2	4
2	2	1	->	0	6	2	5
2	2	2	->	0	0	2	2
2	2	3	->	0	4	2	3
2	3	0	->	0	7	6	4
2	3	1	->	0	3	6	5
2	3	2	->	0	5	6	2
2	3	3	->	0	1	6	3
3	0	0	->	7	2	7	0
3	0	1	->	7	6	7	1
3	0	2	->	7	0	7	6
3	0	3	->	7	4	7	7
3	1	0	->	7	7	3	0
3	1	1	->	7	3	3	1
3	1	2	->	7	5	3	6
3	1	3	->	7	1	3	7
3	2	0	->	1	2	7	4
3	2	1	->	1	6	7	5
3	2	2	->	1	0	7	2
3	2	3	->	1	4	7	3
3	3	0	->	1	7	3	4
3	3	1	->	1	3	3	5
3	3	2	->	1	5	3	2
3	3	3	->	1	1	3	3

Figure 24: Evolution of the $(4, 1/2)$ reversible one dimensional cellular automaton rule $5F0A5F0A$

The connectivity relation associated with this automaton is the following:

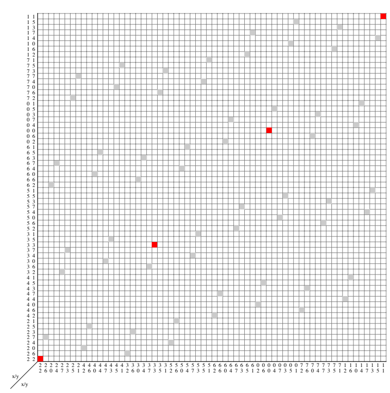


Figure 25: Connectivity relation of the $(4, 1/2)$ reversible one dimensional cellular automaton rule $5F0A5F0A$, the dark points show fixed configurations

The transitive closure of connectivity relation associated with this automaton is the following:

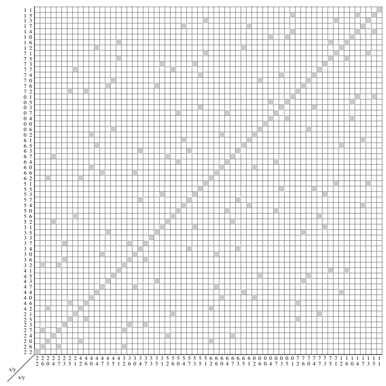


Figure 26: Transitive closure of the connectivity relation of the $(4, 1/2)$ reversible one dimensional cellular automaton rule $5F0A5F0A$

If we rearrange this transitive closure, we obtain the following classes:

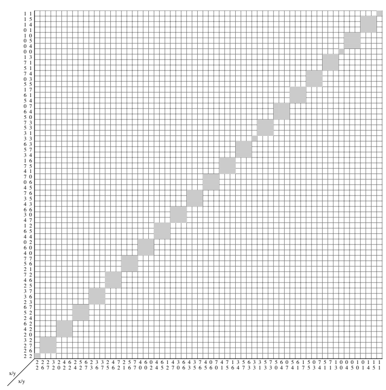


Figure 27: Rearrange of the transitive closure of the connectivity relation of the $(4, 1/2)$ automaton rule $5F0A5F0A$

In this case we obtain the same equivalence class that in the example of section 7.1, therefore both automata belongs to the same dynamical class. Take the block $3, 0$ representing the sequence of states 102. This block has period 3, so the configuration formed with repetitions of the sequence 102 must have period 3, or period 6 in the evolution of the automaton.

An example of this periodical behavior is the following:

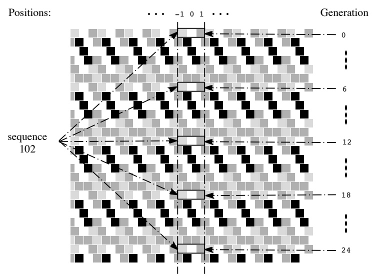


Figure 28: Period 6 corresponding to a period 3 using the composition of the evolution rule in the initial configuration formed with repetitions of the sequence 102

Now, we will see all the possible mappings among sequences of 3 cells using the process described in section 5.4. For example, the mapping of 203 is the following:

Sequence		2	0	3		
p1			6	4		
		0	6	4	0	
		4	6	4	1	
		5	6	4	6	
		1	6	4	7	
		2	6	4	4	
		6	6	4	5	
		7	6	4	2	
p2		3	6	4	3	
	0	0	2	0	1	0
	0	1	2	0	1	1
	1	0	2	0	1	2
	1	1	2	0	1	3
	2	0	2	0	3	0
	2	1	2	0	3	1
	3	0	2	0	3	2
Sequences	3	1	2	0	3	3
		2	0	2	7	
		2	5	2	3	
		3	0	2	5	
		3	5	2	1	
		6	0	4	7	
		6	5	4	3	
		7	0	4	5	
p1		7	5	4	1	
			0	4		
			0	2		
			5	4		
p2			5	2		
		0	2	0		
		0	2	2		
		1	2	0		
Sequences		1	2	2		

Figure 29: All the possible mappings from the sequence 203

Calculating all the mappings among sequences of 3 cells, we have the following mapping among centered cylinder sets:

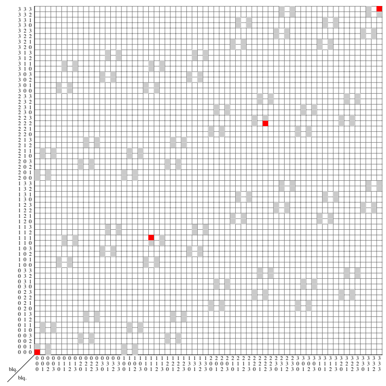


Figure 30: Mapping among centered cylinder sets, the dark points indicate recurrent centered cylinder sets

The transitive closure of the mapping among centered cylinder sets is the following:

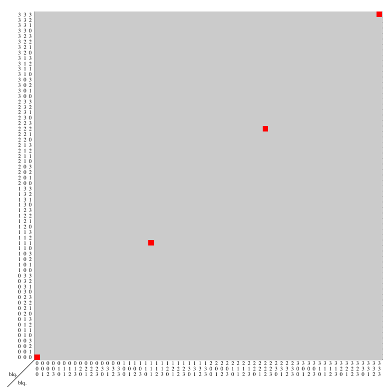


Figure 31: Transitive closure of the mapping among centered cylinder sets, the dark points indicate recurrent centered cylinder sets

Since we only have one equivalence class and there exists centered cylinder sets that can be fixed, then this automaton has topologically mixing orbits. For example, we can form an orbit from the centered cylinder set $\mathcal{C}_{[120]}$ to the centered cylinder set $\mathcal{C}_{[003]}$ in 6 steps, corresponding to 12 evolutions because the composition of the evolution rule. We use the recurrent centered cylinder set $\mathcal{C}_{[111]}$ for constructing such an orbit.

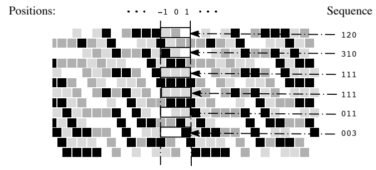


Figure 32: Orbit from the centered cylinder set $\mathcal{C}_{[120]}$ to the centered cylinder set $\mathcal{C}_{[003]}$ in 6 steps

But, since the centered cylinder set $\mathcal{C}_{[111]}$ can be fixed, we can use it to get an orbit from the centered cylinder set $\mathcal{C}_{[120]}$ to the centered cylinder set $\mathcal{C}_{[003]}$ in 7 steps.

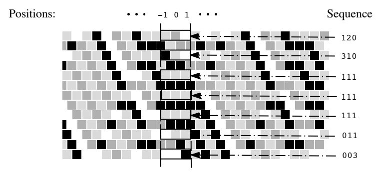
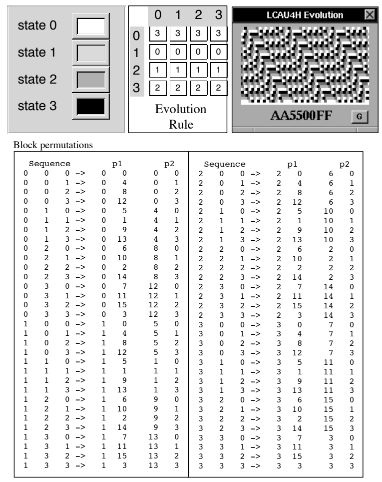


Figure 33: Orbit from the centered cylinder set $\mathcal{C}_{[120]}$ to the centered cylinder set $\mathcal{C}_{[003]}$ in 7 steps

7.3 $(4, 1/2)$ reversible cellular automaton, rule AA5500FF

This automaton has Welch indices $L = 1$ and $R = 4$. The evolution rule, an example of the evolution, and its block permutations are the following:



	0	1	2	3
0	3	3	3	3
1	0	0	0	0
2	1	1	1	1
3	2	2	2	2

Sequence				p1		p2	
0	0	0	->	0	0	0	0
0	0	1	->	0	4	0	1
0	0	2	->	0	8	0	2
0	0	3	->	0	12	0	3
0	1	0	->	0	5	4	0
0	1	1	->	0	1	4	1
0	1	2	->	0	9	4	2
0	1	3	->	0	13	4	3
0	2	0	->	0	6	8	0
0	2	1	->	0	10	8	1
0	2	2	->	0	2	8	2
0	2	3	->	0	14	8	3
0	3	0	->	0	7	12	0
0	3	1	->	0	11	12	1
0	3	2	->	0	15	12	2
0	3	3	->	0	3	12	3
1	0	0	->	1	0	5	0
1	0	1	->	1	4	5	1
1	0	2	->	1	8	5	2
1	0	3	->	1	12	5	3
1	1	0	->	1	5	1	0
1	1	1	->	1	1	1	1
1	1	2	->	1	9	1	2
1	1	3	->	1	13	1	3
1	2	0	->	1	6	9	0
1	2	1	->	1	10	9	1
1	2	2	->	1	2	9	2
1	2	3	->	1	14	9	3
1	3	0	->	1	7	13	0
1	3	1	->	1	11	13	1
1	3	2	->	1	15	13	2
1	3	3	->	1	3	13	3
2	0	0	->	2	0	6	0
2	0	1	->	2	4	6	1
2	0	2	->	2	8	6	2
2	0	3	->	2	12	6	3
2	1	0	->	2	5	10	0
2	1	1	->	2	1	10	1
2	1	2	->	2	9	10	2
2	1	3	->	2	13	10	3
2	2	0	->	2	6	2	0
2	2	1	->	2	10	2	1
2	2	2	->	2	2	2	2
2	2	3	->	2	14	2	3
2	3	0	->	2	7	14	0
2	3	1	->	2	11	14	1
2	3	2	->	2	15	14	2
2	3	3	->	2	3	14	3
3	0	0	->	3	0	7	0
3	0	1	->	3	4	7	1
3	0	2	->	3	8	7	2
3	0	3	->	3	12	7	3
3	1	0	->	3	5	11	0
3	1	1	->	3	1	11	1
3	1	2	->	3	9	11	2
3	1	3	->	3	13	11	3
3	2	0	->	3	6	15	0
3	2	1	->	3	10	15	1
3	2	2	->	3	2	15	2
3	2	3	->	3	14	15	3
3	3	0	->	3	7	3	0
3	3	1	->	3	11	3	1
3	3	2	->	3	15	3	2
3	3	3	->	3	3	3	3

Figure 34: Evolution of the $(4, 1/2)$ reversible one dimensional cellular automaton rule $AA5500FF$

The connectivity relation associated with this automaton is the following:

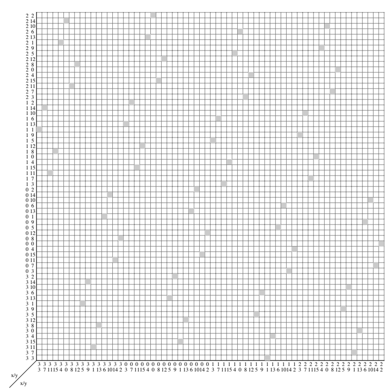


Figure 35: Connectivity relation of the $(4, 1/2)$ reversible one dimensional cellular automaton rule $AA5500FF$

The transitive closure of connectivity relation associated with this automaton is the following:

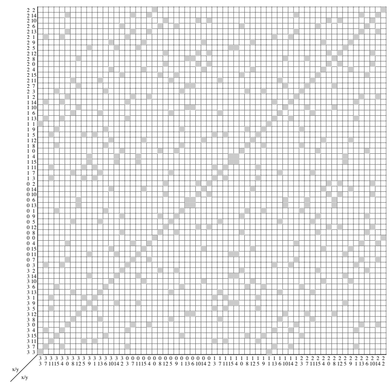


Figure 36: Transitive closure of the connectivity relation of the $(4, 1/2)$ reversible one dimensional cellular automaton rule $AA5500FF$

We have the following classes:

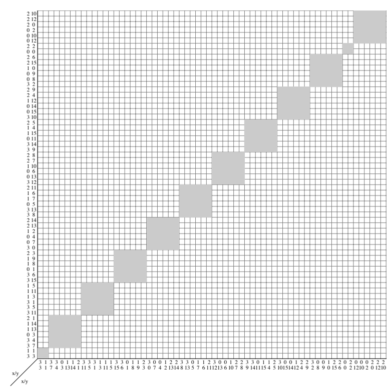


Figure 37: Classes of the transitive closure of the connectivity relation of the $(4, 1/2)$ automaton rule $AA5500FF$

In this case we have 10 classes of 6 elements and 2 classes of 2 elements. Take the block $0, 11$ representing the sequence of states 102. This block has period 6, so the configuration formed with repetitions of the sequence 102 must have period 6, or period 12 in the evolution of the automaton.

An example of this periodical behavior is the following:

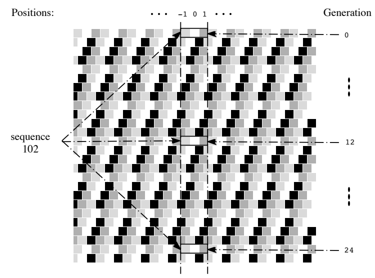


Figure 38: Period 12 corresponding to a period 6 using the composition of the evolution rule in the initial configuration formed with repetitions of the sequence 102

Now, we will see all the possible mappings among sequences of 3 cells using the process described in section 5.4. For example, the mapping of 203 is the following:

Sequence		2	0	3		
p1			1	15		
		0	1	15	0	
		8	1	15	1	
		12	1	15	2	
		4	1	15	3	
		9	1			
		1	1			
		13	1			
		5	1			
		10	1			
		14	1			
		2	1			
		6	1			
		7	1			
		11	1			
		15	1			
p2		3	1			
	0	0	1	3	2	0
	0	1	1	3	2	1
	0	2	1	3	2	2
	0	3	1	3	2	3
	1	0	1			
	1	1	1			
	1	2	1			
	1	3	1			
	2	0	1			
	2	1	1			
	2	2	1			
	2	3	1			
	3	0	1			
	3	1	1			
	3	2	1			
Sequences	3	3	1			
		3	7	2	5	
		3	0	2	9	
		3	9	2	1	
		3	10	2	13	
		0	7			
		0	0			
		0	9			
		0	10			
		1	7			
		1	0			
		1	9			
		1	10			
		2	7			
		2	0			
		2	9			
p1		2	10			
			0	2		
			9	2		
			10	2		
p2			7	2		
		0	0	2		
		1	0	2		
		2	0	2		
Sequences		3	0	2		

Figure 39: All the possible mappings from the sequence 203

Calculating all the mappings among sequences of 3 cells, we have the following mapping among centered cylinder sets:

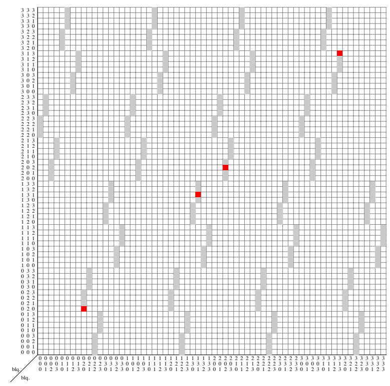


Figure 40: Mapping among centered cylinder sets, the dark points indicate recurrent centered cylinder sets

The transitive closure of the mapping among centered cylinder sets is the following:

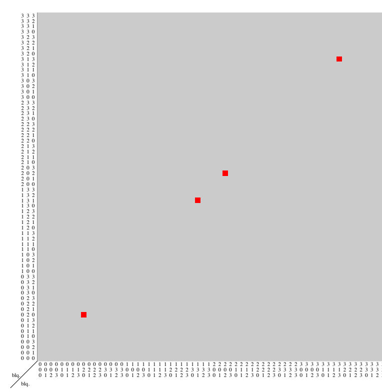


Figure 41: Transitive closure of the mapping among centered cylinder sets, the dark points indicate recurrent centered cylinder sets

Since we only have one equivalence class and there exists centered cylinder sets that can be fixed, then this automaton has topologically mixing orbits. For example, we can form an orbit from the centered cylinder set $\mathcal{C}_{[011]}$ to the centered cylinder set $\mathcal{C}_{[030]}$ in 6 steps, corresponding to 12 evolutions because the composition of the evolution rule. We use the recurrent centered cylinder set $\mathcal{C}_{[020]}$ for constructing such an orbit.

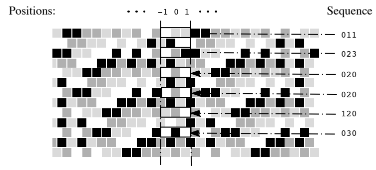


Figure 42: Orbit from the centered cylinder set $\mathcal{C}_{[011]}$ to the centered cylinder set $\mathcal{C}_{[030]}$ in 6 steps

Since the centered cylinder set $\mathcal{C}_{[020]}$ can be fixed, we can use it to get an orbit from the centered cylinder set $\mathcal{C}_{[011]}$ to the centered cylinder set $\mathcal{C}_{[030]}$ in 7 steps.

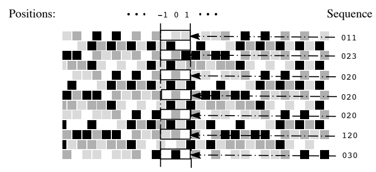
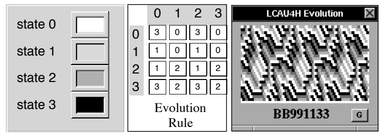


Figure 43: Orbit from the centered cylinder set $\mathcal{C}_{[011]}$ to the centered cylinder set $\mathcal{C}_{[030]}$ in 7 steps

7.4 $(4, 1/2)$ reversible cellular automaton, rule BB991133

This automaton has Welch indices $L = 2$ and $R = 2$. The evolution rule, an example of the evolution, and its block permutations are the following:



	0	1	2	3
0	3	0	3	0
1	1	0	1	0
2	1	2	1	2
3	3	2	3	2

Evolution Rule

Block permutations

Sequence				p1		p2	
0	0	0	->	3	3	0	0
0	0	1	->	3	0	0	5
0	0	2	->	3	7	0	2
0	0	3	->	3	4	0	7
0	1	0	->	4	5	0	4
0	1	1	->	4	0	0	1
0	1	2	->	4	1	0	6
0	1	3	->	4	4	0	3
0	2	0	->	3	5	4	0
0	2	1	->	3	6	4	5
0	2	2	->	3	1	4	2
0	2	3	->	3	2	4	7
0	3	0	->	4	3	4	4
0	3	1	->	4	6	4	1
0	3	2	->	4	7	4	6
0	3	3	->	4	2	4	3
1	0	0	->	5	3	5	0
1	0	1	->	5	0	5	5
1	0	2	->	5	7	5	2
1	0	3	->	5	4	5	7
1	1	0	->	0	5	1	4
1	1	1	->	0	0	1	1
1	1	2	->	0	1	1	6
1	1	3	->	0	4	1	3
1	2	0	->	5	5	1	0
1	2	1	->	5	6	1	5
1	2	2	->	5	1	1	2
1	2	3	->	5	2	1	7
1	3	0	->	0	3	5	4
1	3	1	->	0	6	5	1
1	3	2	->	0	7	5	6
1	3	3	->	0	2	5	3

Sequence				p1		p2	
2	0	0	->	1	3	6	0
2	0	1	->	1	0	6	5
2	0	2	->	1	7	6	2
2	0	3	->	1	4	6	7
2	1	0	->	6	5	6	4
2	1	1	->	6	0	6	1
2	1	2	->	6	1	6	6
2	1	3	->	6	4	6	3
2	2	0	->	1	5	2	0
2	2	1	->	1	6	2	5
2	2	2	->	1	1	2	2
2	2	3	->	1	2	2	7
2	3	0	->	6	3	2	4
2	3	1	->	6	6	2	1
2	3	2	->	6	7	2	6
2	3	3	->	6	2	2	3
3	0	0	->	7	3	3	0
3	0	1	->	7	0	3	5
3	0	2	->	7	7	3	2
3	0	3	->	7	4	3	7
3	1	0	->	2	5	7	4
3	1	1	->	2	0	7	1
3	1	2	->	2	1	7	6
3	1	3	->	2	4	7	3
3	2	0	->	7	5	7	0
3	2	1	->	7	6	7	5
3	2	2	->	7	1	7	2
3	2	3	->	7	2	7	7
3	3	0	->	2	3	3	4
3	3	1	->	2	6	3	1
3	3	2	->	2	7	3	6
3	3	3	->	2	2	3	3

Figure 44: Evolution of the $(4, 1/2)$ reversible one dimensional cellular automaton rule $BB991133$

The connectivity relation associated with this automaton is the following:

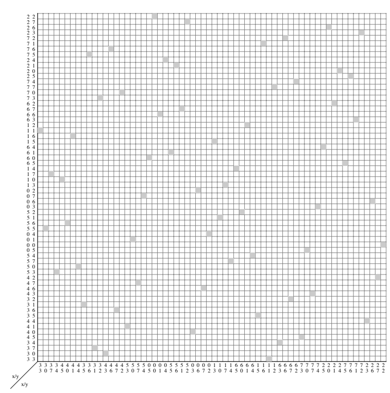


Figure 45: Connectivity relation of the $(4, 1/2)$ reversible one dimensional cellular automaton rule $BB991133$

The transitive closure of connectivity relation of this automaton is the following:

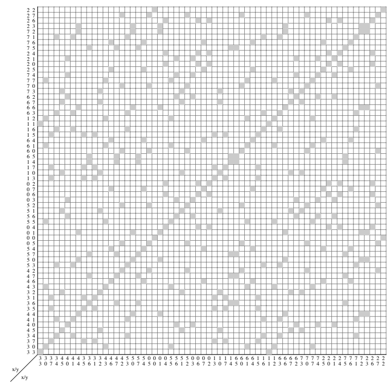


Figure 46: Transitive closure of the connectivity relation of the $(4, 1/2)$ reversible one dimensional cellular automaton rule $BB991133$

If we rearrange this transitive closure, we obtain the following classes:

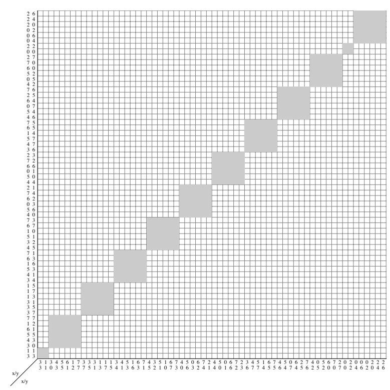


Figure 47: Classes of the transitive closure of the connectivity relation of the $(4, 1/2)$ automaton rule $BB991133$

In this case we obtain the same equivalence class that in the example of section 7.3, therefore both automata belongs to the same dynamical class. Take the block 5, 7 representing the sequence of states 102. This block has period 6, so the configuration formed with repetitions of the sequence 102 must have period 6, or period 12 in the evolution of the automaton.

An example of this periodical behavior is the following:

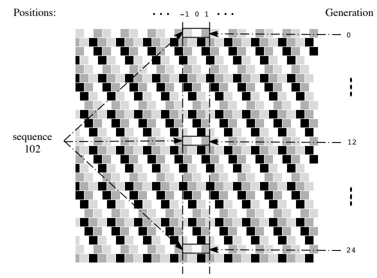


Figure 48: Period 12 corresponding to a period 6 using the composition of the evolution rule in the initial configuration formed with repetitions of the sequence 102

Now, we will see all the possible mappings among sequences of 3 cells using the process described in section 5.4. For example, the mapping of 203 is the following:

Sequence			2	0	3	
p1			1	4		
		0	1	4	0	
		4	1	4	5	
		1	1	4	2	
		5	1	4	7	
		6	1	4	4	
		2	1	4	1	
		7	1	4	6	
p2		3	1	4	3	
	0	1	1	0	2	0
	0	3	1	0	2	1
	1	1	1	0	2	2
	1	3	1	0	2	3
	2	1	1	0	3	0
	2	3	1	0	3	1
	3	1	1	0	3	2
Sequences	3	3	1	0	3	3
		4	0	3	5	
		4	6	3	6	
		0	0	3	1	
		0	6	3	2	
		6	0	4	3	
		6	6	4	6	
		2	0	4	7	
p1		2	6	4	2	
			0	4		
			0	3		
			6	4		
p2			6	3		
			0	1	0	
			0	1	3	
			2	1	0	
Sequences			2	1	3	

Figure 49: All the possible mappings from the sequence 203

Calculating all the mappings among sequences of 3 cells, we have the following mapping among centered cylinder sets:

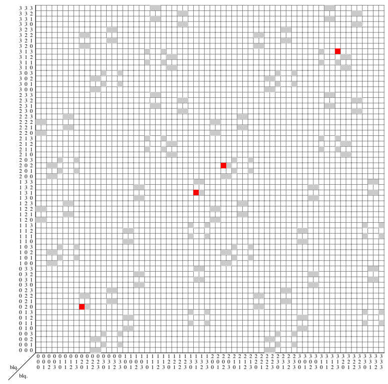


Figure 50: Mapping among centered cylinder sets, the dark points indicate recurrent centered cylinder sets

The transitive closure of the mapping among centered cylinder sets is the following:

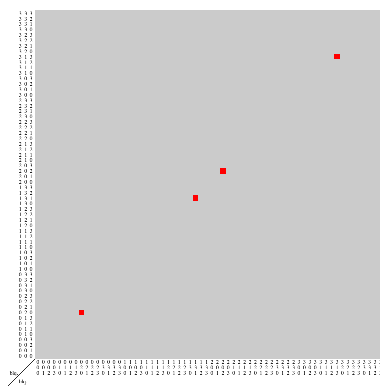


Figure 51: Transitive closure of the mapping among centered cylinder sets, the dark points indicate recurrent centered cylinder sets

Since we only have one equivalence class and there exists centered cylinder sets that can be fixed, then this automaton has topologically mixing orbits. For example, we can form an orbit from the centered cylinder set $\mathcal{C}_{[303]}$ to the centered cylinder set $\mathcal{C}_{[111]}$ in 6 steps, corresponding to 12 evolutions because the composition of the evolution rule. We use the recurrent centered cylinder set $\mathcal{C}_{[131]}$ for constructing such an orbit.

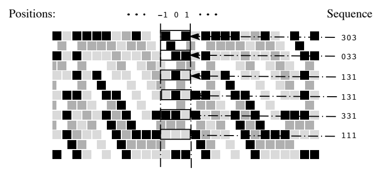


Figure 52: Orbit from the centered cylinder set $\mathcal{C}_{[303]}$ to the centered cylinder set $\mathcal{C}_{[111]}$ in 6 steps

Since the centered cylinder set $\mathcal{C}_{[131]}$ can be fixed, we can use it to get an orbit from the centered cylinder set $\mathcal{C}_{[303]}$ to the centered cylinder set $\mathcal{C}_{[111]}$ in 7 steps.

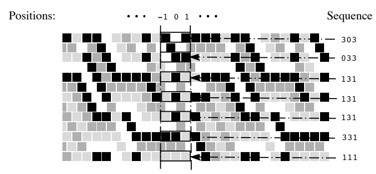


Figure 53: Orbit from the centered cylinder set $\mathcal{C}_{[303]}$ to the centered cylinder set $\mathcal{C}_{[111]}$ in 7 steps

8 Conclusions

The topology of centered cylinder sets and block permutations, give us a way for knowing and classifying different kinds of basic dynamical behaviors in reversible one dimensional cellular automata. We have used very simple matrix methods for finding periodical and transitive behavior in such systems.

As we said at the end of section 5, this matrix methods detect the existence of such behaviors, but they don't show an explicit example of every behavior. The classification proposed in this paper is for automata whose invertible evolution rules have the same nighborhood size, and we have used the representation of any reversible one dimensional cellular automaton with another with neighborhood size equal 2.

This causes that the number of states has a considerably grow. In this way, these methods are easy for computing if the number of states is small.

Experimental observations show that Welch indices are not fundamental for establishing that a given reversible one dimensional cellular automaton belongs to a particular dynamical class. As we see in section 7, a same class has automata with different Welch indices.

Until now, all the automata generated in experimental observations are topologically ergodic and topologically mixing, that is, there exists orbits from every centered cylinder set to all the others. This could be explained by the action of the shift between block permutations, this shift allows that a centered cylinder set can reach a bigger number of centered cylinder sets.

In $(4, 1/2)$ reversible one dimensional cellular automata, a preliminary examination shows only 2 kinds of dynamical classes. The first classification has 24 classes, 20 with 3 elements and 4 with 1 element each one. The second classification has 12 classes, 10 of 6 elements and 2 with 2 elements. Another question is which is the influence of the uniform multiplicity both in connectivity and transition relations, i.e., in which way the uniform multiplicity establishes the quantitative behavior of the connections that every centered cylinder set has to.

9 List of symbols

Numbers

$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integer numbers
$\mathbb{Z}^+$	integer positive numbers

Elements of cellular automata

K	set of states
k	number of states
c	configuration
r	neighborhood radius
φ	local rule evolution
φ^{-1}	inverse local rule of φ
K^{2r+1}	family of neighborhoods
C	set of configurations
Φ	mapping between configurations induced by φ
Φ^{-1}	inverse mapping of Φ

Sequences of elements of K

w	state sequence
K^*	set of finite state sequences
$[a_i, a_f]$	sequence beginning in position a_i and ending in position a_f

Centered cylinder Set

$\mathcal{C}_{[w]}$	cylinder set centered in sequence w
$\mathfrak{C}$	family of centered cylinder sets

Generic sets and generic spaces

X	generic set
x	generic element
$\cap$	intersection of sets
$\mathcal{O}$	open set
d	generic distance
$d(x_1, x_2)$	distance between elements x_1 and x_2
Ψ	generic global mapping
(X, Ψ)	generic dynamical system

Configuration space

$(C, \mathfrak{C})$	topological configuration space
$\varphi \circ \varphi$	composition of evolution rules
e	configuration sequence
(C, Φ)	dynamical system of cellular automata

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Glossary

block permutation Permutation from a sequence of $6r$ cells to another sequence with the same length. This permutation is represented by a pair xy and is used for characterizing reversible one dimensional cellular automata. *page 12*

cellular automaton Discrete dynamical system formed with discrete states. Its behavior depends on local interactions among their components. *page 7*

configuration Inital sequence of states or sequence of states produced by the evolution of a one dimensional cellular automaton. *page 7*

connectivity relation Relation among block permutations that shows their periodical behavior. *page 20*

centered cylinder set Set of configurations that have the same central finite sequence of states. *page 9*

dynamical system System that changes in time due to the action of a global mapping. *page 15*

evolution rule Mapping from neighborhoods to states in a one dimensional cellular automaton. *page 7*

evolution of cellular automata Mapping from a configuration to another established by the global mapping induced by an evolution rule. *page 7*

fixed point Point that remains without change under the iteration of the global mapping. *page 15*

neighborhood Sequence of cells evolving to a new cell. *page 7*

neighborhood radius Number of cells at each side of every cell that form a neighborhood. *page 7*

neighborhood size Size of a neighborhood in a one dimensional cellular automata. *page 7*

non-wandering set Open set with a point whose orbit returns to the same open set in a finite number of steps. *page 15*

NXLCAU (NeXT Linear Cellular Automata) System developed by Harold V. McIntosh using the operating system NeXT. This system calculates and provides a great number of tools for analyzing one dimensional cellular automata. *page 25*

orbit The trajectory described by a given point in a dynamical system under the iteration of the global mapping. *page 15*

periodic point Point with an orbit that returns to itself in a finite number of steps. *page 15*

reversible cellular automaton Cellular automaton whose global mapping is invertible. *page 7*

RLCAU (Reversible Linear Cellular Automata) Based on NXLCAU, this system calculates $(k, 1/2)$ reversible one dimensional cellular automata using block permutations. Also gives some tools for studying such systems. *page 25*

transitive closure Apply the transitive property to a given relation. This can be done using the Warshall algorithm. *page 20*

topologycally ergodic Open set with a point which orbit reachs all the others open sets. *page 16*

topologycally mixing Open set with a point which orbit reachs any open set and it can remain there for undefined steps. *page 16*

transitive point Point with an orbit that reachs all the other open sets. *page 16*

transition relation Relation among centered cylinder sets induced by block permutations. This relation shows the transitive behavior of the centered cylinder sets defined by sequences of length $6r$ cells. *page 21*

uniform multiplicity Property of surjective one dimensional cellular automata, it defines that every finite sequence have k^{2r} ancestors. *page 12*

Welch indices Indices that represent the number of different extensions in both sides of the ancestors in reversible one dimensional cellular automata. *page 12*