On Complex Numbers

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Abstract

An introduction to general aspects of the theory of complex variable, we provide the complex number definition, its various representations and finally gives a brief introduction to complex functions.

1 Introduction.

Have you ever wondered how the product of two equal numbers is never negative ? Now and then the solution of equations leads to try to find the square root of negative numbers, but you should know that the product of two equal numbers can not be negative. Then, this hardly seems have a solution. Fortunately there is an invention to solve this problem. An approach is to consider a number such that its square is minus one. But there are no such real numbers in the real numbers field. Now, consider a Real numbers field as a line whose points are ordered. On left-hand side are negatives, on right-hand side are positives and there is the nought on the middle of line. When you multiply a number by -1, what you are really doing is to rotate π radians counterclockwise about the origin of the real axis. When you multiplied by -1 again, you rotates once again and returns to the original point, *i.e.,* finally you are rotate 2π radians counterclockwise about origin real axis. In fact, any number can be multiplied by $(-1)^2$ without being modified. Now, is there any number that multiplied by itself remains at -1? Notwithstanding how much you try to find such number you will notice that there is no real number that satisfies this restriction. Jean-Robert Argand found a way to solve that, if the multiplication by -1 rotates π radians about the origin of real axis and multiplying $(-1)^2$ returns to original number, then, there exists a number called i, such that $i^2 = -1$ hence i means rotate $\pi/2$ radians counterclockwise about the origin of the real line. On the other hand, this means that such new number i lies outside the real line. Indeed this allows leaving the real axis and come into a plane called the complex plane or Argand's plane.

The number $i = \sqrt{-1}$ is called *imaginary* and has some interesting particularities. If i is squared $i^2 = -1$ and if else is cubed $i^3 = -i$ and even more $i^4 = 1$ and $i^5 = -1$ and so on. Every positive integer k such that i^k fall into four possible values $(1, i, -1, -i)$ as show in the following table:

i^k	k						sequence
1	0	4	8	12	...	n	$4n$
i	1	5	9	13	...	n	$4n+1$
-1	2	6	10	14	...	n	$4n+2$
$-i$	3	7	11	15	...	n	$4n+3$

In this sense, may be possible to reduce any i^k to one of four options by matching k with the sequence it belongs to. Thus it holds that the remainder after dividing by k is zero. For instance, we want to calculate i^{20}, then it is necesary to find the sequence that divides 20 leaving zero as remainder, comparing with the first sequence $20 = 4n$ we see that 20 mod $4 = 0$ hence $i^{20} = 1$. Even more, k could be greater *e.g.,* i^{65535}, in this case it is the same procedure, $65535 = 4n$ but now $65535 \bmod 4 = 3$ and $n \notin \mathbb{Z}$, for this reason $i^{65535} \neq 1$ then we need to compare $65535 = 4n + 1$ and so $65534 \bmod 4 = 2$ therefore $n \notin \mathbb{Z}$, again $i^{65535} \neq i$, now we going to compare $65535 = 4n + 2$ then $65533 \bmod 4 = 1$ so that $i^{65535} \neq -1$, finally we get $65535 = 4n + 3$ then $65532 \bmod 4 = 0$ and $n \in \mathbb{Z}$, we have found that $i^{65535} = -i$. Thus it is posible to reduce i^k where $k \in \mathbb{Z}_+$.

2 Overview.

We define a complex number as follows:

$$z = x + iy \tag{1}$$

Where:

x Is the real part of z *i.e.,* $x \in \mathbb{R}$

iy Represents the imaginary part of z where $y \in \mathbb{R}$ and $i \in \mathbb{I}$

z Is the complex number, where $z \in \mathbb{C}$

Complex numbers are a combination of both real and imaginary parts, these numbers include the field of real numbers *i.e.,* the field of real numbers is a subfield of the field of complex numbers. An other way of representing a complex number could be (x, y). We will now define a couple of complex numbers $z_1 = a + ib$ and $z_2 = c + id$ and their properties are:

$$\begin{aligned} z_1 = z_2 &\leftrightarrow \{a = c \text{ and } b = d\} \\ z_1 + z_2 &= (a + c) + i(b + d) \\ z_1 z_2 &= (ac - bd) + i(ad + bc) \\ \overline{z_1} &= a - ib \end{aligned} \tag{2}$$

$\overline{z_1}$ is called complex conjugate and represents a reflection on the real axis, it also has some particularities:

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2} \\ \overline{z_1 \cdot z_2} &= \overline{z_1} \cdot \overline{z_2} \end{aligned}$$

Now, we may define the complex division as well:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{z_1}{z_2} \cdot \frac{\overline{z_2}}{\overline{z_2}} \\ \frac{z_1}{z_2} &= \left(\frac{a + ib}{c + di}\right)\left(\frac{c - id}{c - id}\right) \\ \frac{z_1}{z_2} &= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} \end{aligned}$$

To understand these concepts we present the follwing simple example:

$$\begin{aligned} z_1 &= 2 + i1.5 \\ z_2 &= -1 + i2.4 \end{aligned}$$

This numbers could be represented as points on the complex plane:

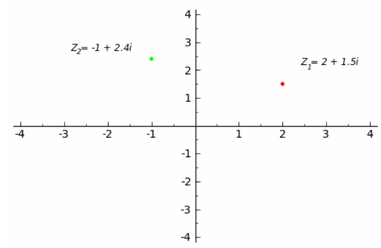

Figure 1: You only need a number for representing a point on the complex plane.

Now, we going to show some operations with them. The complex conjugate:

$$\begin{aligned} \overline{z_1} &= 2 - i1.5 \\ \overline{z_2} &= -1 - i2.4 \end{aligned}$$

Complex addition:

$$\begin{aligned} z_1 + z_2 &= (2 + i1.5) + (-1 + i2.4) \\ z_1 + z_2 &= (2 - 1) + i(1.5 + 2.4) \\ z_1 + z_2 &= 1 + i3.9 \end{aligned}$$

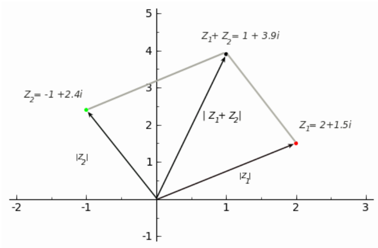

Figure 2: Complex addition represents the triangle inequality.

Complex multiplication and division:

$$z_1 z_2 = (2 + i1.5)(-1 + i2.4)$$
$$z_1 z_2 = ((2)(-1) - (1.5)(2.4)) + i((2)(2.4) + (-1)(2.5))$$
$$z_1 z_2 = -5.6 + i3.3$$

$$\frac{z_1}{z_2} = \Big(\frac{2 + i1.5}{-1 + i2.4}\Big)\Big(\frac{-1 - i2.4}{-1 - i2.4}\Big)$$
$$\frac{z_1}{z_2} = \frac{(-2 + 3.6) + i(-4.8 - 1.5)}{1 + 5.76}$$
$$\frac{z_1}{z_2} = \frac{1.6 - i6.3}{6.76}$$
$$\frac{z_1}{z_2} = 0.23668639053 - i0.93195266272$$

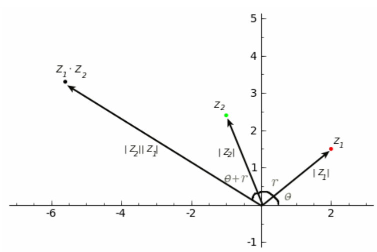

Figure 3: In complex multiplication, modulus of numbers are multiplied and their arguments are added.

You can see that there are a few terms which were not defined: The modulus $|z|$ and argument z_θ which sometimes is called the *phase*. Modulus of a complex number is a positive real number. $|z|$ measure the length of z to origin. The term *modulus* was introduced by Jean-Robert Argand and it is defined as:

$$|z| = |x + iy| = \sqrt{x^2 + y^2}$$

Geometrically speaking, a modulus of the difference of two complex numbers is the distance between two points. If only one of them is $z_0 = 0 + i0$ then the modulus is the distance of a point to the origin. For instance, $|z| = |\frac{1}{2} + i\frac{\sqrt{3}}{2}| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$. Particulary this is a norm, so that complex plane is a normalized vectorial space of finite dimension, hence is a metric space whose metric is given by the application:

$$\mathbb{C} \times \mathbb{C} \to \mathbb{R}_+, (z_1, z_2) \mapsto |z_2 - z_1|$$

Where $|z_2 - z_1|$ is a distance. Modulus has the following properties:

$$|z| \geq 0$$
$$|z| = 0 \leftrightarrow z = 0$$
$$|z_1 + z_2| \leq |z_1| + |z_2|$$
$$|z_1 - z_2| \geq ||z_1| - |z_2||$$
$$|z_1 z_2| = |z_1||z_2|$$
$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$$
$$|\bar{z}| = |z| = |-\bar{z}| = |-z|$$

$$z \cdot \bar{z} = |z|^2 \tag{3}$$

Turning back to our example, we could know the modulus and argument of the complex numbers defined previously as follows:

$$|z_1| = \sqrt{4 + 2.25} = \sqrt{6.25} = 2.5$$
$$|z_2| = \sqrt{1 + 5.76} = \sqrt{6.76} = 2.6$$
$$|z_1 + z_2| = \sqrt{1 + 15.21} = \sqrt{16.21} = 4.02616$$
$$|z_1| + |z_2| = 2.5 + 2.6 = 5.1$$
$$|z_1 z_2| = \sqrt{31.36 + 10.89} = \sqrt{42.25} = 6.5$$
$$|z_1||z_2| = (2.5)(2.6) = 6.5$$
$$\left|\frac{z_1}{z_2}\right| = \left|\frac{1.6 - i6.3}{6.76}\right| = 0.96153846$$
$$\frac{|z_1|}{|z_2|} = \frac{2.5}{2.6} = 0.96153846$$

Now let is check that the arguments of two complex numbers are added when the numbers are multiplied. To do this, we will define θ_1 and θ_2 as the argument of z_1 and z_2 respectively and φ as the argument of $z_1 z_2$.

$$\begin{aligned}
&\theta_1 = \tan^{-1}\left(\tfrac{1.5}{2}\right) = 0.643501108 = 36.8^\circ \\
&\theta_2 = \tan^{-1}\left(\tfrac{2.4}{-1}\right) = -1.176005207 = -67.38^\circ \\
&180^\circ + \theta_2 = 112.6^\circ \\
&\varphi = \tan^{-1}\left(\tfrac{3.3}{-5.6}\right) = \tan^{-1}(-0.589285714) \\
&\varphi = -0.532504098 = -30.51^\circ \\
&180^\circ + \varphi = 149.4^\circ \\
&36.8^\circ + 112.6^\circ = 149.4^\circ \\
&0.64350110 + (-1.17600520) = -0.53250409
\end{aligned}$$

Now we know the concept of modulus and argument, then we may also define a complex number as:

$$z = |z|\mathrm{e}^{i\theta} \tag{4}$$

Both representations are equivalent, *i.e.*, this definition is equivalent to the equation defined by (1), to verify this we will prove the equation defined by (3).

$$\begin{aligned}
z\bar{z} &= |z|^2 \\
z\bar{z} &= |z|^2\mathrm{e}^{i(\theta-\theta)} \\
z\bar{z} &= |z|^2\mathrm{e}^{i\theta}\mathrm{e}^{-i\theta} \\
z\bar{z} &= |z|\mathrm{e}^{i\theta}|z|\mathrm{e}^{-i\theta} \\
z\bar{z} &= |z||\bar{z}|
\end{aligned}$$

Wich is the same as:

$$\begin{aligned}
z\bar{z} &= |z|^2 \\
z\bar{z} &= (\sqrt{x^2+y^2})^2 \\
z\bar{z} &= x^2 + y^2 \\
z\bar{z} &= x^2 + y^2 + i(xy - xy) \\
z\bar{z} &= x^2 - ixy + ixy + y^2 \\
z\bar{z} &= (x+iy)(x-iy) \\
z\bar{z} &= |z||\bar{z}|
\end{aligned}$$

3 Some interesting features.

We know that a complex number can be represented as a modulus(distance) and a argument(angle), so you could think that there is a manner of rotate a complex number by increasing the angle, this is correct. Moreover, it is possible rotate a complex number only using real numbers. If you multiply a complex number by i, you rotate it $\pi/2$ radians counterclockwise, this rotation and can be expressed as $z \cdot i = (x+iy)(i) = -y+ix$ in a matrix representation:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

Now, we consider a complex number which is inscribed on the unit circle and has a little increment on imaginary axis *i.e,* $w = 1 + i\epsilon, \epsilon > 0$. Whenever any complex number is multiplied by w, undergoes a little rotation counterclockwise, but ϵ should be consider very small to minimize any increase in the modulus.

$$\begin{aligned}
z \cdot w &= (x+iy)(1+i\epsilon) \\
z \cdot w &= (x - y\epsilon) + i(y + x\epsilon)
\end{aligned}$$

Also can be written in matrix representation:

$$\begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y\epsilon \\ y + x\epsilon \end{pmatrix} \texttt{Small rotation.}$$

Which is a matrix whose elements consist entirely of real numbers. By this way, you can rotate a complex number. If you want a greater rotation, you should multiply $\left(\begin{smallmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{smallmatrix}\right)$ again.

$$\begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix}\begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \texttt{Two small rotations.}$$

For instance, we want to rotate the complex number $z = 1+i2$ where $x = 1, y = 2$ and we will to consider $\epsilon = 0.1$, then we get the following outcomes when we multiplies $\begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ many times. :

n	x	y
1	1.9	1.2
2	1.78	1.39
3	1.641	1.568
4	1.4842	1.7321
5	1.31099	1.88052
6	1.122938	2.011619
7	0.9217761	2.1239128

These numbers are showed at fig.4, you can notice that the z point is rotating counterclockwise according to be multiplied by the real matrix.

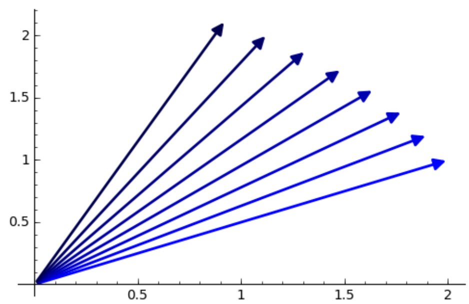

Figure 4: Rotating a complex number by means of multiplying by a Real matrix.

This way of rotating a complex number has a drawback when ϵ is not considered very small, recalling complex multiplication definition of two numbers we get that both modulus are multiplied too, then obviously $|(1+i\epsilon)| > 1$ when $\epsilon > 0$, therefore any complex number that is multiplied undergoes a tiny increment at its modulus. It seems that is necesary to find a number such that has unit modulus. We could find such a number by rotating $z = 1 + i0$ counterclockwise. This is nothing but multiply the vector $(1, 0)$ by 2D rotation matrix as follows:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Althought it is yields a two dimensional vector $(\cos\theta, \sin\theta)$, may be represented by the complex number $w = \cos\theta + i\sin\theta$ which has a unit modulus, therefore any complex number which is multiplied by w remains its length and undergoes a rotation of θ about the origin. Regardless that the symbol θ is often used to represent an angle in degrees, we use it to represent an angle in radians. Now, we going to represent it according to the equation (4):

$$w = \cos\theta + i\sin\theta = e^{i\theta}$$

Inasmuch as we know that $|w| = 1$, *i.e.*, w is the point on the unit circle at θ angle, then w may be used as *bona fide* rotation operator, furthermore it makes way for one of the most interesting asseverations of this document:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

In spite of the fact that we defined a complex number $z = |z|e^{\theta}$ out of the blue prevously, and now, we define that $e^{i\theta} = \cos\theta + i\sin\theta$, is worth the trouble to make a formal demonstration. So that, recalling from Taylor's Theorem, which, when is centered at the origin, is sometimes called Maclaurin's series, we get some trigonometrical functions called exponential, cosine and sine, which are represented by power series as follows:

$$\begin{aligned} e^{x} &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \end{aligned}$$

Each of these functions $f(x)$ must be an infinitely differentiable function, whose values and values of all of its derivatives, exist at zero. Perhaps you already could figure out that there is a relationship between the equations described above, nevertheless is not entirely clear. Now, putting x equal to real value θ in both sine and cosine functions and replacing x by $i\theta$ in the exponential function it seems that exhibits a closer relation.

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \\ \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \end{aligned}$$

Therefore:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{5}$$

The equation (5) have some interesting features. One of them is that $\sin\pi = 0$ and $\cos\pi = -1$. This fact was discovered by Leonhard Euler around 1740, and it is called *Euler's Formula* in his honour. Now, it looks almost obvious:

$$e^{i\pi} + 1 = 0$$

This amazing formula exhibits a remarkable connection between the numbers $e, i, \pi, 1$ and 0. This kind of stuff is one of the reasons why the mathematics are beautiful.

4 Complex functions

Some of the properties of a real function are shown when is drawn. In case of complex function is not straightforward because the resulting values are not on a line, but on a plane. For this reason, it is feasible to draw both planes separately. Whilst functions of real numbers are a cartesian product of two planes which yields a two dimensional graph, the complex function representation requires a cartesian product of two planes, hence it is necesary some way to represent a four dimensional graph or simply show both planes separately. Obviously is more widely acepted the second option. This special case of depict functions is called *maping*. In the meantime, we define a complex function as the cartesian product of two complex planes that takes a complex number as argument and produces another complex number which has real functions as coefficients. Formally speaking:

$$f : \mathbb{C} \times \mathbb{C} \mapsto \mathbb{C}$$

Do not forget that a complex number has two real numbers representing both real and imaginary parts, therefore:

$$\begin{aligned} z &= x + iy \\ f(z) &= a + ib \\ a &= u(x, y) \\ b &= v(x, y) \end{aligned}$$

To explain this we will see the following example:

$$\begin{aligned} z &= x + iy \\ f(z) &= z^2 \\ f(z) &= (x + iy)^2 \\ f(z) &= x^2 + 2ixy - y^2 \\ f(z) &= (x^2 - y^2) + i(2xy) \\ f(z) &= u(x, y) + iv(x, y) \\ u(x, y) &= x^2 - y^2 \\ v(x, y) &= 2xy \end{aligned}$$

Now and then it is commonly wondering which numbers ought to be considered for be maped by the function. The figure 5 shows a centered grid of complex numbers, each number is represented as a point in the plane and all of they are linked together by forming a rectangular grid.

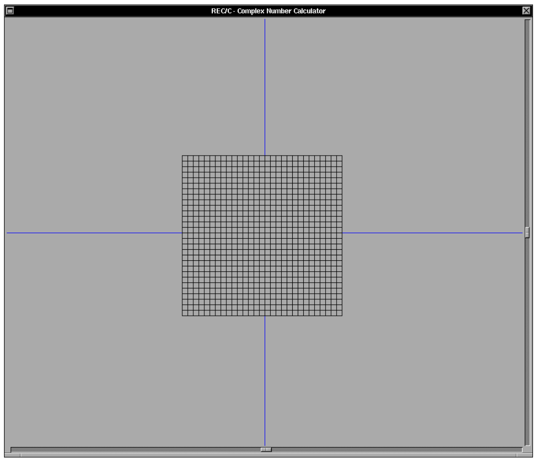

Figure 5: Centered grid of complex numbers

Now, it is necesary to evaluate each and every point of the grid (*i.e.*, a complex number) and plot it on a new complex plane. In this manner you can notice that a complex function is nothing but a transformation which takes a complex number on the plane and takes it to another place on the same plane. For ilustrative purposes we consider $f(z) = z^2$. Thus, the complex number $4 + i2$ become $12 + i16$ and so on. We will apply this function to our grid and pay atention to its consecuences.

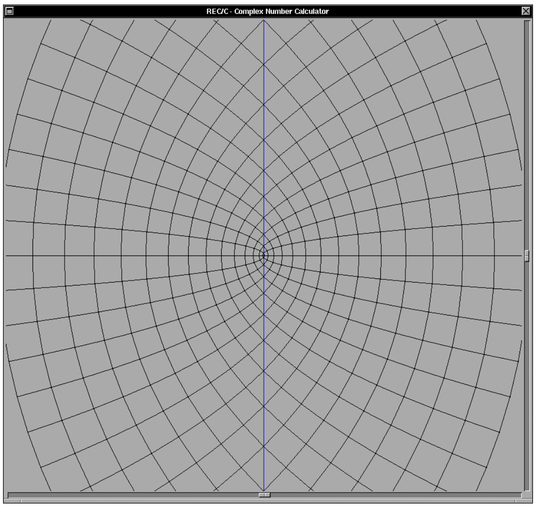

Figure 6: Complex squared $f(z) = z^2$ REC-C code: (PP*;) f

We use a comptuer software for ploting complex functions named REC-C which was created by Dr. Harold V. McIntosh for the NextStep Operating System. If you have a chance of get a copy, this is our code template:

```
{ (P;) f   [ complex function:  f(z)  ]
  (Z QB $-100,0$ Gng  Y* Gng  Qk;) g   [draw axis subroutine]
(  $100$Mp     [ set  canvas size to 100 ]
  @g            [draw axis ]
  $-1.5,-1.5$  (!30! P @f G p (!30! @f g G p u+ ;;) p v+ ;;)   [ draw horizontal lines ]
  $-1.5,-1.5$  (!30! P @f G p (!30! @f g G p v+ ;;) p u+ ;;)   [ draw vertical lines ]
;)}
```

Figure 7: Code Template for the REC-C examples

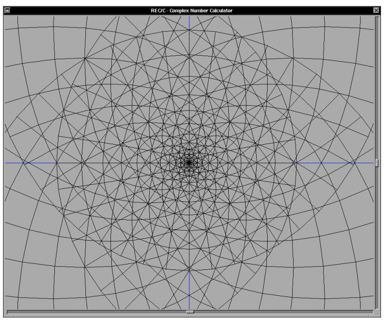

Figure 8: Complex cubed $f(z) = z^3$ REC-C code: (PPP**;) f

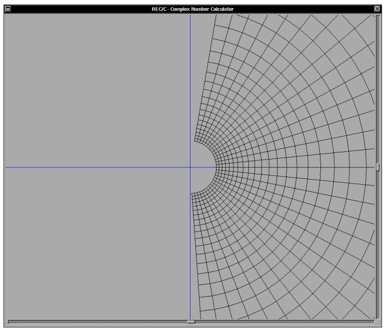

Figure 9: Complex exponential $f(z) = e^z$ REC-C code: (PE;) f

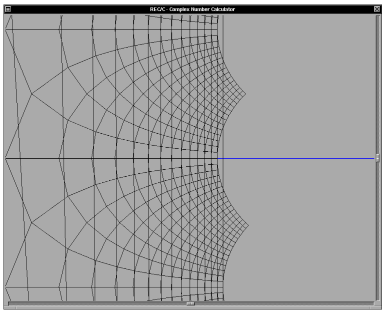

Figure 10: Complex loagarithm $f(z) = Log(z)$ REC-C code: (PL;) f

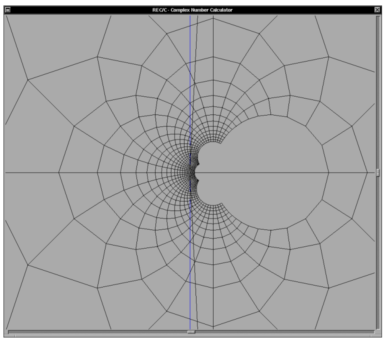

Figure 11: Complex fractional linear $f(z) = \dfrac{z+1}{z-1}$ REC-C code: (PF;) f or (PX+PXD-/;) f

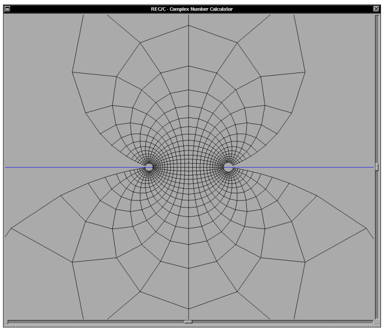

Figure 12: Complex hyperbolic tangent $f(z) = \tanh z$ REC-C code: (PT;) f

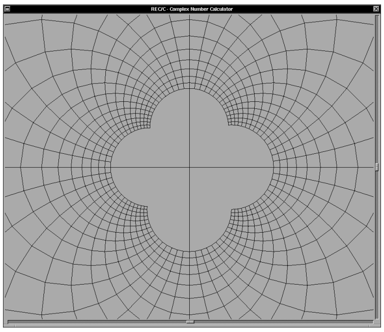

Figure 13: Complex Inverse $f(z) = \frac{1}{z}$ REC-C code: (PX&/;) f

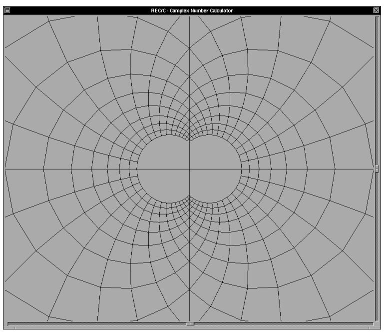

Figure 14: Complex Inverse Squared $f(z) = \frac{1}{z^2}$ REC-C code: (PP*X&/;) f

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